Gravitational Lenses

[Schneider, Ehlers, & Falco, *Gravitational Lenses,* Springer-Verlag 1992]

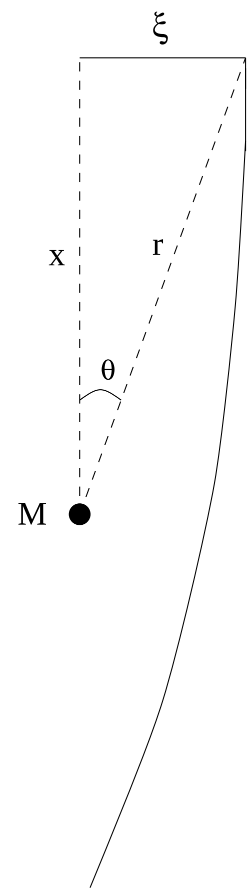
Consider a photon moving past a point of mass, $\mathcal{M}$, with an starting "impact parameter," ξ. From classical Newtonian gravity, the photon will undergo an acceleration perpendicular to the direction of its motion. Under the Born approximation, the amount of this deflection can be calculated simply by integration along the path.

$$\frac{dv_\perp}{dt} = \frac{G\mathcal{M}}{r^2} \sin\theta \qquad (21.01)$$

If we substitute $dx = c\,dt$, then

$$v_\perp = \frac{G\mathcal{M}}{c} \int_{-\infty}^{\infty} \frac{1}{x^2+\xi^2} \cdot \frac{\xi}{(x^2+\xi^2)^{1/2}}\, dx$$

$$= \frac{G\mathcal{M}\xi}{c} \int_{-\infty}^{\infty} \left(x^2+\xi^2\right)^{-3/2} dx \quad (21.02)$$



The integral in (21.02) is analytic, and works out to $2/\xi^2$. So

$$v_\perp = \frac{2G\mathcal{M}}{\xi c} \qquad (21.03)$$

and the Newtonian deflection angle (in the limit of a small deflection) is

$$\alpha = \frac{v_\perp}{c} = \frac{2G\mathcal{M}}{\xi c^2} \qquad (21.04)$$

In the general relativistic case, however, gravity affects both the spatial and time component of photon's path, so the actual bending is twice this value. Thus, we define the angle of deflection, otherwise known as the Einstein angle, as

$$\alpha = \frac{4G\mathcal{M}}{\xi c^2} \tag{21.05}$$

To understand the geometry of a gravitational lens, let's first define a few terms. Let

$$\begin{aligned}
D_d &= \text{distance from the observer to the lens} \\
D_s &= \text{distance from the observer to the light source} \\
D_{ds} &= \text{distance from the lens to the source} \\
\vec{\beta} &= \text{true angle between the lens and the source} \\
\vec{\theta} &= \text{observed angle between the lens and the source.} \\
\vec{\xi} &= \text{distance from the lens to a passing light ray} \\
\vec{\alpha} &= \text{the Einstein angle of deflection}
\end{aligned}$$

Note that β, θ, α, and ξ are all vectors, and calculations must deal with negative, as well as positive angles. For a simple point-source lens, the vectorial components of the angles make no difference, but in systems where the lens is complex (*i.e.*, several galaxies/clusters along the line-of-sight) the deflection angles must be added vectorially.

In practice, α, β, and θ are all very small, so small angle approximations can be used. Also, D_d, D_s, and D_{ds} are all much bigger than ξ. Thus, the deflection can be considered instantaneous (*i.e.*, we can use a "thin lens" approximation).

The geometry of a simple gravitational lens system is laid out in the figure. From $\triangle OSI$ and the law of sines,

$$\frac{\sin(180-\alpha)}{D_s} = \frac{\sin(\theta-\beta)}{D_{ds}} \qquad (21.06)$$

Since all the angles are small, $\sin(\theta-\beta) \approx \theta-\beta$, and $\sin(180-\alpha) = \sin\alpha \approx \alpha$. So

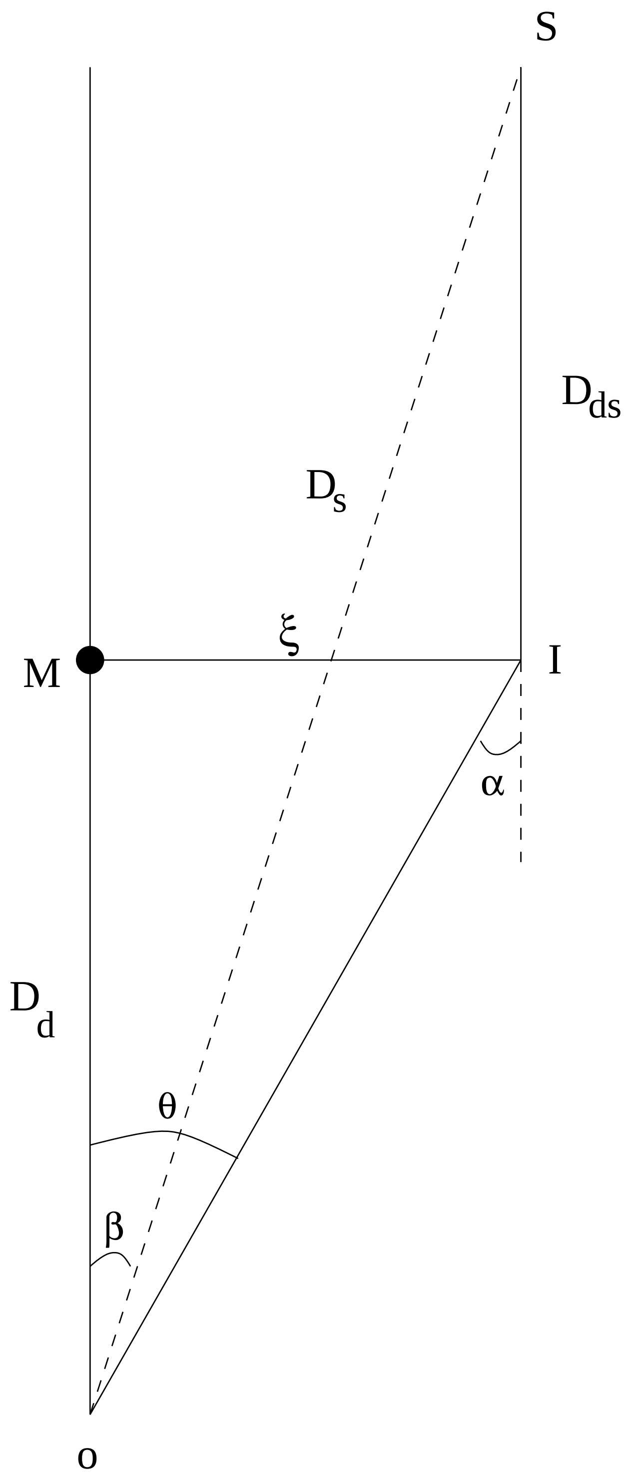
$$\vec{\beta} = \vec{\theta} - \frac{D_{ds}}{D_s}\vec{\alpha} \qquad (21.07)$$

(Again, for single point lenses, the fact that the angles are vectors are irrelevant. For simplicity, I will therefore drop the vector signs for the rest of these derivations.)

Since we have already seen that

$$\alpha = \frac{4G\mathcal{M}}{\xi c^2} \qquad (21.05)$$

and, from the simple geometry of small angles, $\theta = \xi/D_d$,



$$\beta = \theta - \left(\frac{4G\mathcal{M}}{c^2} \frac{D_{ds}}{D_d D_s} \right) \cdot \frac{1}{\theta} \tag{21.08}$$

In other words, we have a relation between β and θ. But note: for a given value of β, there is more than one value of θ that will satisfy the equation. This is a general theorum of lenses. For non-transparent lenses (such as the Schwarzschild lens being considered) there will always be an even number of images; for transparent lenses, there is always an odd number of images.

To simplify, let's define the characteristic bending angle, α_0, as a quantity that depends only on the mass of the lens and the distances involved

$$\alpha_0 = \left(\frac{4G\mathcal{M}}{c^2} \frac{D_{ds}}{D_d D_s} \right)^{1/2} \tag{21.09}$$

so that

$$\beta = \theta - \frac{\alpha_0^2}{\theta} \tag{21.10}$$

If β and θ are co-planar (as they will be for a point source lens), then (21.10) is equivalent to the quadratic equation

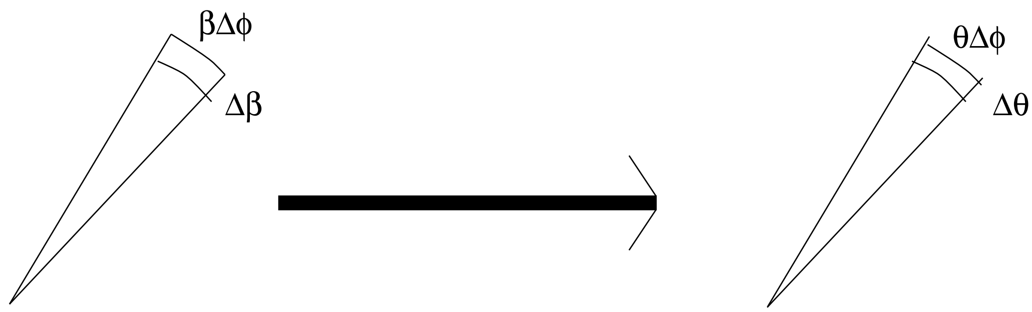
$$\theta^2 - \beta\theta - \alpha_0^2 = 0 \tag{21.11}$$

whose solution is

$$\theta = \frac{1}{2} \left(\beta \pm \sqrt{4\alpha_0^2 + \beta^2} \right) \tag{21.12}$$

This defines the location of the gravitional lens images as a function of the true position of the source with respect to the lens, β, and α_0.

Magnification for a Schwarzschild Lens



The relation

$$\theta = \frac{1}{2}\left(\beta \pm \sqrt{4\alpha_0^2 + \beta^2}\right) \tag{21.12}$$

implies that at least one image will be magnified. To see this, let's lay out a polar coordinate system with the lens at the center, and consider the light passing through a differential area, dA. At the lens, this area element is $dA = \beta\Delta\phi\,\Delta\beta$. However, due to the gravitational lens, the angles are distorted, so that the observer gets sees this light squeezed into area $dA' = \theta\Delta\phi\,\Delta\theta$. Thus, the lens has "focussed" the light from area dA to area dA'. The magnification will therefore be the ratio of the two areas

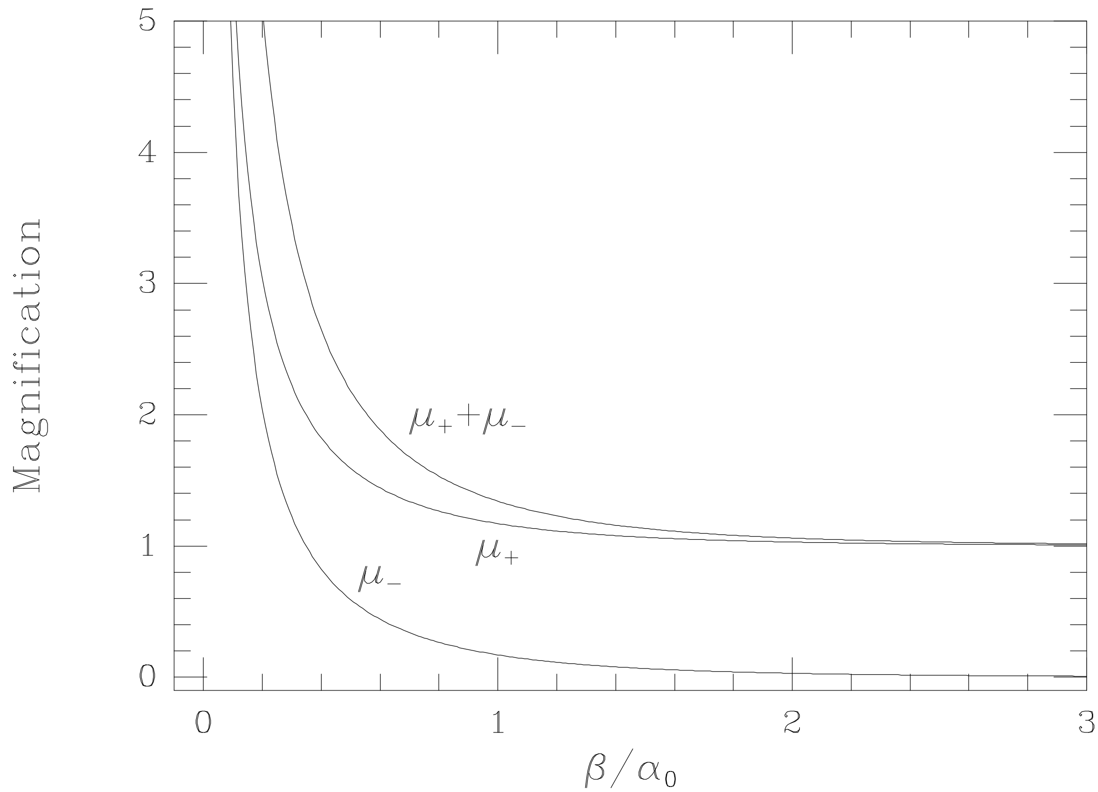
$$\mu = \left|\frac{\theta\Delta\phi\,\Delta\theta}{\beta\Delta\phi\Delta\beta}\right| = \left|\frac{\theta}{\beta}\frac{\Delta\theta}{\Delta\beta}\right| = \left|\frac{\theta}{\beta}\frac{d\theta}{d\beta}\right| \tag{21.13}$$

The last term is just the derivative of (21.12), or

$$\frac{d\theta}{d\beta} = \frac{1}{2}\left(1 \pm \frac{\beta}{\sqrt{4\alpha_0^2 + \beta^2}}\right) \tag{21.14}$$

which means that the magnification is

$$\mu = \frac{1}{4}\left\{\frac{\beta}{\left(4\alpha_0^2 + \beta^2\right)^{1/2}} + \frac{\left(4\alpha_0^2 + \beta^2\right)^{1/2}}{\beta} \pm 2\right\} \tag{21.15}$$



Note the limits: as $\beta \longrightarrow \infty$, $\mu_+ \longrightarrow 1$ and $\mu_- \longrightarrow 0$; in other words, there is no lensing. On the other hand, when $\beta \longrightarrow 0$, both μ_+ and μ_- go to infinity. As you can see from the curves, when $\beta < \alpha_0$, gravitational lens magnification is important.

Another feature of this magnification is that it only occurs because the solid angle through which the light is flowing is being distorted. Moreover, nothing happens in the ϕ direction; it is only the θ coordinate that gets compressed. Consequently, the shape of the source will be distorted. The generalized version of (21.13) is

$$\frac{1}{\mu} = \frac{(\Delta\omega)_0}{\Delta\omega} = \left| \det \frac{\partial \beta}{\partial \theta} \right| = \frac{A_s}{A_I} \left(\frac{D_d}{D_s} \right)^2 \tag{21.16}$$

where ω_0 is the solid angle subtended by the source in the absense of lensing, and amplification is given by the determinant of the Jacobian matrix. Note that this Jacobian can either be larger than one (magnification) or smaller than one (de-magnification). It can also be positive or negative; this gives the image's orientation with respect to the original source. (This is also known as the parity.) A general theorum of transparent lenses is that there is always at least one image with positive parity and magnification.

Finally, note that determinant in the above equation can be zero. (Or, equivalently, the magnification given in (21.15) can be infinite. These locations are called *caustics.* In practice, the magnification never gets to be infinite because a) the background sources have finite size, and b) if it ever does get near infinite, interference effects will become important.

There are a few other relations that should be kept in mind. Since the magnification of a lens is given by

$$\mu = \frac{1}{4}\left\{\frac{\beta}{\left(4\alpha_0^2+\beta^2\right)^{1/2}} + \frac{\left(4\alpha_0^2+\beta^2\right)^{1/2}}{\beta} \pm 2\right\} \tag{21.15}$$

the brightness ratio of two images is

$$\nu = \frac{\mu_+}{\mu_-} = \left\{\frac{\left(4\alpha_0^2+\beta^2\right)^{1/2}+\beta}{\left(4\alpha_0^2+\beta^2\right)^{1/2}-\beta}\right\}^2 \tag{21.17}$$

which simplies through (21.12) to

$$\nu = \frac{\mu_+}{\mu_-} = \left(\frac{\theta_1}{\theta_2}\right)^2 \tag{21.18}$$

Some other convenient relations (which can be proved via simple algebra) are

$$\theta_1 - \theta_2 = \left(4\alpha_0^2+\beta^2\right)^{1/2} \tag{21.19}$$

$$\beta = \theta_1 + \theta_2 \tag{21.20}$$

and

$$\theta_1\theta_2 = -\alpha_0^2 \tag{21.21}$$

(Again, remember, that the angles are vector quantities, so angles such as θ can be positive or negative.)

General Scale Lengths for Lenses

It's useful to keep in mind some of the numbers that are typical of gravitational lens problems. Remember that the key value is α_0,

$$\alpha_0 = \left(\frac{4G\mathcal{M}}{c^2} \frac{D_{ds}}{D_d D_s} \right)^{1/2} \tag{21.09}$$

For the MACHO project, which tries to detect stars in the Large Magellanic Cloud ($D_s = 50$ kpc) that are lensed by stellar-type objects in the Milky Way,

$$\alpha_0 = 4 \times 10^{-4} \left(\frac{\mathcal{M}}{\mathcal{M}_\odot} \right)^{1/2} \left(\frac{D_{ds}}{D_d} \right)^{1/2} \text{ arcsec} \tag{21.22}$$

In this case, the image separations $\theta \sim \alpha_0$ are sub-milliarcsec, so you will never be able to resolve the two separately lensed images. But, with both images landing on top of each other, the source can be (greatly) magnified.

For quasar lensing by a typical large elliptical galaxy ($10^{12}\mathcal{M}_\odot$) in an Einstein-de Sitter universe,

$$\alpha_0 = 1.6 \left(\frac{\mathcal{M}}{10^{12}\mathcal{M}_\odot} \right)^{1/2} \left(\frac{z_s - z_d}{z_s z_d} \right)^{1/2} h^{1/2} \text{ arcsec} \tag{21.23}$$

where h is the Hubble Constant in units of 100 km/s/Mpc. Here, the typical scale-length is a little over an arcsec, so two separate images can be resolved.

Gravitational Lens Time Delay

If the source behind a gravitational lens is variable, then, of course, the images we see will also be variable. However, the images will not necessarily vary at the same time: in general, there will be a *time delay* between the two events. This comes from two components.

First, there is a purely geometrical time delay. Consider the value of $D_{ds} + D_d - D_s$ for each image of the lens. From $\triangle OSI$ and the law of cosines

$$D_s^2 = D_d^2 + D_{ds}^2 - 2D_d D_{ds} \cos(180 - \alpha) \tag{21.24}$$

If we substitute

$$\cos(180 - \alpha) = -\cos\alpha = -\left\{1 - 2\sin^2(\alpha/2)\right\} \tag{21.25}$$

then

$$D_s^2 = D_d^2 + D_{ds}^2 + 2D_d D_{ds} - 4D_d D_{ds} \sin^2(\alpha/2) \tag{21.26}$$

or

$$\begin{aligned} 4D_d D_{ds} \sin^2(\alpha/2) &= \left(D_d^2 + D_{ds}^2\right)^2 - D_s^2 \\ &= (D_d + D_{ds} - D_s)(D_d + D_{ds} + D_s) \end{aligned} \tag{21.27}$$

If we now use the law of sines,

$$\frac{\sin\alpha}{D_s} = \frac{\sin(\theta - \beta)}{D_{ds}} \tag{21.28}$$

and consider that all the angles are small, so that $\sin\alpha \approx \alpha$, $\sin(\theta - \beta) \approx \theta - \beta$, $\sin^2 \alpha/2 \approx \alpha^2/4$, and $D_d + D_{ds} + D_s \approx 2D_s$, then

$$D_d + D_{ds} - D_s = \frac{4D_d D_{ds} \sin^2(\alpha/2)}{D_d + D_{ds} + D_s}$$
$$= \frac{D_d D_{ds} \alpha^2}{2D_s} \qquad (21.29)$$

Finally, by substituting for α using (21.07), we have

$$D_d + D_{ds} - D_s = \frac{D_d D_s}{2D_{ds}} \left(\vec{\theta} - \vec{\beta}\right)^2 = c\,\Delta t \qquad (21.30)$$

Since there are two values of $\vec{\theta}$, there are two values for the geometrical time delay. So the delay of one image with respect to the other (which is all we can observe), is

$$\Delta T_g = \frac{D_d D_s}{2cD_{ds}} \left\{(\theta_1 - \beta)^2 - (\theta_2 - \beta)^2\right\} \qquad (21.31)$$

If you now recall that

$$\alpha_0 = \left(\frac{4G\mathcal{M}}{c^2} \frac{D_{ds}}{D_d D_s}\right)^{1/2} \qquad (21.09)$$

and

$$\beta = \theta - \frac{\alpha_0^2}{\theta} \qquad (21.10)$$

then, we can substitute these in to get

$$\Delta T_g = \frac{1}{2c} \frac{4G\mathcal{M}}{c^2 \alpha_0^2} \left\{\frac{\alpha_0^4}{\theta_1^2} - \frac{\alpha_0^4}{\theta_2^2}\right\}$$
$$= \frac{2G\mathcal{M}}{c^3} \alpha_0^2 \left\{\frac{\theta_2^2 - \theta_1^2}{\theta_1^2 \theta_2^2}\right\} \qquad (21.32)$$

But, we also have the relation

$$\theta = \frac{1}{2}\left(\beta \pm \sqrt{4\alpha_0^2 + \beta^2}\right) \tag{21.12}$$

which gives

$$\theta_1\theta_2 = -\alpha_0^2 \tag{21.21}$$

So, with this substitution, we get the final expression for the geometrical time delay from one image to the other.

$$\Delta T_g = \frac{2G\mathcal{M}}{c^3}\left\{\frac{\theta_1^2 - \theta_2^2}{|\theta_1\theta_2|}\right\} \tag{21.33}$$

The second source of signal time delay arises from the passage of the photons through a potential. Any time light passes through a gravitational potential, it experiences the Shapiro time delay

$$\Delta t = -\frac{2}{c^3}\int \Phi dl \tag{21.34}$$

where dl is the photon's path through the potential well. For the case of a weak potential and small deflection, the delay a photon experiences going from source S to point I is

$$\begin{aligned} \Delta t_1 &= -\frac{2}{c^3}\int_0^{D_{ds}} \frac{G\mathcal{M}}{r} dl = -\frac{2G\mathcal{M}}{c^3}\int_0^{D_{ds}} \frac{dx}{(x^2+\xi^2)^{1/2}} \\ &= -\frac{2G\mathcal{M}}{c^3}\ln\left\{x + \sqrt{x^2+\xi^2}\right\}\Big]_0^{D_{ds}} = \frac{2G\mathcal{M}}{c^3}\ln\left(\frac{\xi}{2D_{ds}}\right) \end{aligned} \tag{21.35}$$

Similary, the delay in going from point I to the observer is

$$\Delta t_2 = \frac{2G\mathcal{M}}{c^3}\ln\left(\frac{\xi}{2D_d}\right) \tag{21.36}$$

So the total time delay for the potential is

$$\Delta t_p = \Delta t_1 + \Delta t_2 = \frac{2G\mathcal{M}}{c^3}\left\{\ln\left(\frac{\xi}{2D_{ds}}\right) + \ln\left(\frac{\xi}{2D_d}\right)\right\} \quad (21.37)$$

We can now calculate the difference in the time delay from one image to the other, by realizing that at very large distances from the lense, the potential delay is negligible. So, if we pick some distance, $D' \gg \xi$, but $D' \ll D_{ds}, D_s$, then we can write (21.37) as

$$\Delta t_p = \frac{2G\mathcal{M}}{c^3}\left\{\ln\left(\frac{\xi}{2D'}\right) + \ln\left(\frac{D'}{D_{ds}}\right) + \ln\left(\frac{\xi}{2D'}\right) + \ln\left(\frac{D'}{D_d}\right)\right\} \quad (21.38)$$

The second and fourth terms do not depend on ξ and are the same for both images. So, when we take the difference of the two time delays we get

$$\Delta T_p = \frac{4G\mathcal{M}}{c^3}\left\{\ln\left(\frac{\xi_i}{2D'}\right) - \ln\left(\frac{\xi_2}{2D'}\right)\right\} \quad (21.39)$$

If we then substitute for ξ using $\theta = \xi/D_d$ and note that D' is the same for both rays, we get the final expression for the difference between the two Shapiro delays

$$\Delta T_p = \frac{4G\mathcal{M}}{c^3}\ln\left(\left|\frac{\theta_1}{\theta_2}\right|\right) \quad (21.40)$$

The total time delay of one image with respect to the other is then

$$\Delta T = \Delta T_g + \Delta T_p = \frac{4G\mathcal{M}}{c^3}\left\{\frac{\theta_1^2 - \theta_2^2}{2|\theta_1\theta_2|} + \ln\left(\left|\frac{\theta_1}{\theta_2}\right|\right)\right\} \quad (21.41)$$

Note, however, that this equation neglects two important effects.

1) The universe is expanding and possibly non-Euclidean. Thus, the expressions for distance depend on q_0, H_0, and z in some complex manner. (They are angular size distances.)
2) Cosmological lenses are not point sources. The best lenses are elliptical galaxies (which are fairly concentrated collections of stars), but these galaxies are located inside of clusters. So real lenses include the effects of both the galaxy and the cluster (which, of course, add vectorially).

The Uses of Gravitational Lenses

Gravitational lenses are extremely interesting for several reasons.

1) Because the magnification properties of lenses depend on the mass of the lensing object, microlensing experiments, such as the MACHO experiment of monitoring millions of stars in the Large Magellanic Cloud, can probe the properties of halo stars (and other objects) that are two faint to see. In fact, by studying the light curve of a gravitational microlens, one can deduce not only the mass of the intervening lens (with some assumption about the lens' distance), but also whether the lens is a single star or something else. Microlensing experiments have measured light curves that must be due to binary-star lenses, and, in theory, it is even possible to deduce the presence of an earth-mass planet circling a lens.

2) Because of their high magnification, gravitational lenses can be used to observe high-redshift galaxies that would normally be much too faint to observe otherwise. There is a slight problem with this, in that the image of the galaxy is usually distorted (spread out) over many pixels (into an arc). However, if the light can be collected, you can use it to study a representative sample of normal (not exceptionally bright) high-redshift galaxies.

3) Because the time delay of a gravitational lens depends on distance, these objects can be used to measure the Hubble Constant. Unfortunately, there are two problems with this. First, the time delay depends not only on the distance to the lens, but its mass as well. This mass is not known *a priori:* for a typical elliptical galaxy lens, one must try to infer the lens mass from the velocity dispersion of its stars. Second, the geometry of the lens is usually complicated. Elliptical galaxies are in clusters, and the cluster's mass will also contribute to the time delay. One must therefore measure the velocity dispersion of the galaxies in the cluster in order to determine the cluster's mass (while, of course, hoping that the cluster is virialized, and that none of the objects you are measuring are foreground or background objects). Moreover, for distant sources, it is likely that the calculations will be further complicated by the presence of multiple lenses along the line-of-sight.

4) All clusters (and halos) along a line-of-sight cause some (small) amount of magnification. But perhaps more importantly, because the mapping of the source into the image plane is asymmetric, it will cause the background images to be distorted. Thus, it is possible to obtain a census of galaxies (and clusters) just be looking for small distortions in the shapes of distant galaxies. By seeing how this (weak) signal changes with position on the sky, one can derive information about galaxy clusters throughout the universe, and constrain the evolution of heirachical clustering (*i.e.*, Λ).

General Comments on Real Lenses

As we have seen, the bending angle for a lens is

$$\alpha = \frac{4G\mathcal{M}}{\xi c^2} \tag{21.05}$$

This implies that as $\xi \longrightarrow 0$, the bending angle (and magnification) becomes infinite. This doesn't happen with cosmological lenses, because the lenses are never point masses. Instead, the mass is distributed, *i.e.,* $\mathcal{M}(\xi)$, which goes to zero as ξ goes to zero. So the equations do not diverge.

In practice, the gravitational deflection due to a galaxy's cluster is almost as important as that of the galaxy itself. Consequently, one must model the lens with a galaxy plus cluster potential, and this breaks the symmetry of the problem. Hence, the equation to solve is

$$\vec{\alpha} = \int \frac{4G \sum \vec{\xi}'}{c^2} \; \frac{\vec{\xi} - \vec{\xi}'}{|\vec{\xi} - \vec{\xi}'|^2} \, d\xi' \tag{21.42}$$

In other words, the bending angle is the vector sum of the gravitational deflections caused by all the mass in the two-dimensional cluster.

The best lenses are massive and compact objects. For cosmological lenses, this means elliptical galaxies. (They are the most compact of all galaxies.) Consequently, even though the majority of galaxies are spirals, it is generally reasonable to assume that all gravitatational lenses are caused by ellipticals. Ideally, to use a gravitational lens as a cosmological probe, you would like

1) Optical identifications for both the lens and the source. (Occasionally, the lens is too faint to be studied.)

2) A measured velocity dispersion for the lens (to estimate its mass).

3) A variable source (to measure the time delay for the two images).

4) An angular separation between the two images of $1'' < \Delta\theta < 2''$. If the separation is any smaller, it is hard to resolve the images; if the separation is any larger, then the cluster's mass begins to dominate the problem. Also, recall

$$\Delta T = \Delta T_g + \Delta T_p = \frac{4G\mathcal{M}}{c^3}\left\{\frac{\theta_1^2 - \theta_2^2}{2|\theta_1\theta_2|} + \ln\left(\left|\frac{\theta_1}{\theta_2}\right|\right)\right\} \tag{21.41}$$

hence the gravitational time delay $\Delta T \propto \Delta\theta^2$. Small separations keep the time delay manageable. (For instance, the lens 0957+561 has a separation of $6''$, and a time delay of between 1 and 2 years.)

5) VLBI radio structure, so that the shape of the images can be used to constrain the mass distribution of the lens.

6) Multiple images to constrain the geometry. Four images are better than two, and an Einstein ring is best.

Statistical Microlensing

Consider a Schwartzschild (point source) lens at redshift z in front of a source at redshift z_s. The magnification of the two images is given by

$$\mu = \frac{1}{4}\left\{\frac{\beta}{\left(4\alpha_0^2+\beta^2\right)^{1/2}} + \frac{\left(4\alpha_0^2+\beta^2\right)^{1/2}}{\beta} \pm 2\right\} \tag{21.15}$$

If the two images are unresolved (*i.e.,* the case of microlensing), then the total magnification is

$$\mu_T = \mu_+ + \mu_- = \frac{1}{2}\left\{\frac{2\alpha_0^2+\beta^2}{\beta\left(4\alpha_0^2+\beta^2\right)^{1/2}}\right\} \tag{21.43}$$

With a little bit of math, this equation can be inverted to give the angle β, which yields a total magnification of μ_T

$$\beta^2 = 2\alpha_0^2\left\{\frac{\mu_T}{\left(\mu_T^2-1\right)^{1/2}} - 1\right\} \tag{21.44}$$

The cross-sectional area for a magnification of greater than μ_T to occur is therefore

$$\sigma(\mu_T, z, z_s) = \pi\beta^2 \tag{21.45}$$

This means that at redshift z_s, any object located in an area of

$$\sigma(\mu_T, z, z_s) = \pi D_s^2\beta^2 \tag{21.46}$$

will be microlensed.

Now, let's calculate the probability that a source at redshift z_s will be microlensed. To begin, let's assume that the density of lenses is given by η. (Since the best lenses are elliptical galaxies, you can consider η to be the density of elliptical galaxies.) For simplicity, let's also assume that η does not evolve with time (there are as many elliptical galaxies today as there were at redshift z), and that each lens has identical lensing properties (same mass, same mass concentration, *etc.*)

If η_0 is the density of lenses today, then the density of lenses at redshift z is given by

$$\eta(z) = \eta_0(1+z)^3 \tag{21.47}$$

and the total number of lenses between z and $z + dz$ is

$$dN(z) = \eta dV = \eta_0(1+z)^3 dV \tag{21.48}$$

where dV is the cosmological volume element. (For Euclidean space, this would normally just be $4\pi r^2 dr$, but because of the universal expansion and the possible non-flat geometry of space, the actual expression is a complicated function of z, q_0, and Λ.)

Now if the density of lenses is small enough so that the probability of two lenses overlapping each other is negligible, then the total area at redshift z_s that is microlensed up to at least a magnification of μ_T is just the sum of the cross-sections of all the lenses, *i.e.*,

$$\sigma_t(z_s) = \int \sigma(\mu_T, z, z_s) dN = \int_0^{z_s} \sigma(\mu_T, z, z_s)\eta_0(1+z)^3 dV \tag{21.49}$$

The probability of a microlens is then just this area, normalized to the total surface area of the universe at redshift z_s

$$A_t = 4\pi D_{ang}^2 \tag{21.50}$$

where D_{ang}^2 is the angular-size distance of redshift z.

This "statistical microlensing" is extremely important. In fact, with typical numbers for the space density of elliptical galaxy lenses, you find that by $z = 2$, half the objects in the sky are magnified by at least 10%!