THE JAFFE LAW

[Jaffe 1983, *MNRAS,* **202,** 995]

A useful relation, which has fallen out of favor over the best decade, is the Jaffe law for the true 3-D space density of an elliptical galaxy.

$$j(r) = \frac{\mathcal{L} r_J}{4\pi r^2 (r + r_J)^2} \tag{A.01}$$

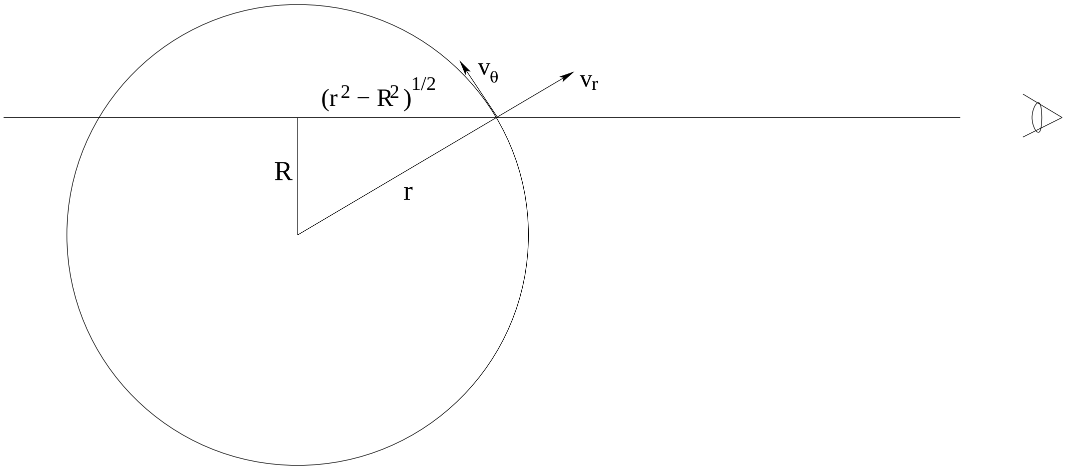
where r_J is the Jaffe radius. This law has a number of advantages. First, it is simple. Second, it implies a total luminosity that is finite and easy to calculate

$$\begin{aligned} \mathcal{L}_T &= \int_0^\infty 4\pi r^2 j(r) dr \\ &= \int_0^\infty 4\pi r^2 \frac{\mathcal{L} r_J}{4\pi r^2 (r + r_J)^2} dr \\ &= \mathcal{L} r_J \int_0^\infty (r + r_J)^{-2} = \mathcal{L} r_J \left(r + r_J\right)^{-1} \Bigg|_0^\infty = \mathcal{L} \quad \text{(A.02)} \end{aligned}$$

The space radius containing 1/2 the light is also simple. From (A.02), the fraction of light contained in radius x is

$$f = \frac{\mathcal{L}(x)}{\mathcal{L}_T} = -\mathcal{L} r_J (r + r_J)^{-1} \Bigg|_0^x \Bigg/ - \mathcal{L} r_J (r + r_J)^{-1} \Bigg|_0^\infty \tag{A.03}$$

For $f = 0.5$, this is easily solved: $x = r_J$. The Jaffe radius is therefore the radius within the galaxy that contains half the light. For reference, r_J is related to the galaxy's effective radius (*i.e.*, the projected radius which encloses half the light by $r_J = 1.3106 R_e$.



To go from true space density to projected density, one must integrate along the line-of-sight

$$I(R) = 2\int_R^\infty j(r)\,\cdot\,\frac{r}{\left(r^2 - R^2\right)^{1/2}}\,dr \tag{A.04}$$

which, for the Jaffe law, is

$$I(R) = \frac{2\mathcal{L}r_J}{2\pi}\int_R^\infty \frac{1}{r^2(r+r_J)^2}\,\cdot\,\frac{r}{\left(r^2 - R^2\right)^{1/2}}\,dr \tag{A.05}$$

This integral is analytic, though its solution is long

$$I(R) = \frac{\mathcal{L}}{r_J^2}\left\{\frac{r_J}{4R} + \frac{1}{2\pi}\left[\frac{r_J^2}{r_J^2 - R^2} - \frac{2r_J^3 - r_J R^2}{\left(r_J^2 - R^2\right)^{3/2}}\operatorname{arccosh}\left(\frac{r_J}{R}\right)\right]\right\} \tag{A.06}$$

for $R < r_J$, and

$$I(R) = \frac{\mathcal{L}}{r_J^2}\left\{\frac{r_J}{4R} + \frac{1}{2\pi}\left[\frac{r_J^2}{R^2 - r_J^2} + \frac{r_J R^2 - 2r_J^3}{\left(R^2 - r_J^2\right)^{3/2}}\arccos\left(\frac{r_J}{R}\right)\right]\right\} \tag{A.07}$$

for $R > r_J$.

While the above properties of Jaffe law are interesting, the true purpose of the expression is for galactic dynamics. For the case of a constant mass-to-light ratio (Υ), the law yields a simple, analytic expression for the galactic potential. If one converts luminosity to mass via $\mathcal{M} = \Upsilon \mathcal{L}$, then the space density of matter in a Jaffe elliptical is law

$$\rho(r) = \frac{\mathcal{M} r_J}{4\pi r^2 \left(r + r_J\right)^2} \tag{A.08}$$

One can plug this into Poisson's equation

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Phi}{dr} \right) = 4\pi G \rho \tag{A.09}$$

to get

$$\frac{d}{dr} \left(r^2 \frac{d\Phi}{dr} \right) = \frac{4\pi G \mathcal{M} r_J r^2}{4\pi r^2 (r + r_J)^2} = \frac{G \mathcal{M} r_J}{\left(r + r_J\right)^2} \tag{A.10}$$

This can simply be integrated to yield

$$r^2 \frac{d\Phi}{dr} = G \mathcal{M} r_J \int \frac{dr}{\left(r + r_J\right)^2} = -\frac{G \mathcal{M} r_J}{\left(r + r_J\right)} + C \tag{A.11}$$

where C is the constant of integration. Since the force at $r = 0$ must be identically zero,

$$\frac{d\Phi}{dr}(r = 0) = -\frac{G\mathcal{M}}{r^2} \left(\frac{r_J}{r + r_J} \right) + \frac{C}{r^2} = 0 \Longrightarrow C = G\mathcal{M} \tag{A.12}$$

Thus, the potential implied by the Jaffe law is

$$\begin{aligned}
\Phi &= \int -\frac{G\mathcal{M}r_J}{r^2\,(r+r_J)} + \frac{G\mathcal{M}}{r^2}\,dr \\
&= \int -\frac{G\mathcal{M}r_J}{r^2\,(r+r_J)} + \frac{G\mathcal{M}(r+r_J)}{r^2(r+r_J)}\,dr \\
&= \int \frac{G\mathcal{M}}{r(r+r_J)}\,dr \\
&= \frac{G\mathcal{M}}{r_J}\ln\left(\frac{r}{r+r_J}\right) \qquad \text{(A.13)}
\end{aligned}$$

Finally, for the case of isotropic or circular orbits, the Jaffe law yields expressions for the stellar velocity dispersions that are analytic. From galactic dynamics, the Jeans equation in spherical coordinates is

$$\frac{\partial\rho\langle v_r^2\rangle}{\partial r} + \frac{\rho}{r}\left\{2\langle v_r^2\rangle - \left(\langle v_\theta^2\rangle + \langle v_\phi^2\rangle\right)\right\} = -\rho\,\frac{\partial\Phi}{\partial r} \qquad \text{(A.14)}$$

where $\langle v_r^2\rangle$, $\langle v_\theta^2\rangle$, and $\langle v_\phi^2\rangle$ are the radial, tangential, and azimuthal stellar velocity dispersions. To simplify the notation, we can define the radial and tangential "pressures" as density-weighted velocity dispersions, *i.e.*,

$$P = \rho\,\langle v_r^2\rangle \qquad Q = \rho\left(\langle v_\theta^2 + \langle v_\phi^2\rangle\right) \qquad \text{(A.15)}$$

In this case, the Jeans equations looks a bit like the equation for hydrostatic equilibrium

$$\frac{dP}{dr} + \frac{(2P-Q)}{r} = -\rho\frac{d\Phi}{dr} \qquad \text{(A.16)}$$

If we substitute in the densities and force law implied by the Jaffe model

$$\begin{aligned}\frac{dP}{dr}+\frac{(2P-Q)}{r} &= -\frac{\mathcal{M}r_J}{4\pi r^2(r+r_J)^2}\cdot\frac{G\mathcal{M}}{r_J}\frac{r}{r+r_J}\cdot\\ &\qquad\left\{\frac{1}{r+r_J}-\frac{r}{(r+r_J)^2}\right\}\\ &= -\frac{G\mathcal{M}^2\,r_J}{4\pi r^3\,(r+r_J)^3}\end{aligned} \tag{A.17}$$

This is a relatively simple expression, which reduces further in the presence of isotropic or circular orbits. In the isotropic case, $\langle v_r^2\rangle=\langle v_\theta^2\rangle=\langle v_\phi^2\rangle$, so $P=Q/2$, and the Jeans equation becomes

$$\frac{dP}{dr}=-\frac{G\mathcal{M}^2\,r_J}{4\pi r^3\,(r+r_J)^3} \tag{A.18}$$

which has the solution

$$\begin{aligned}P &= -\frac{G\mathcal{M}^2\,r_J}{4\pi}\int r^{-3}\,(r+r_J)^{-3}\,dr\\ &= \frac{G\mathcal{M}^2 r_J}{4\pi r^2(r+r_J)^2}\cdot\left\{\frac{r_J^3-2r_J^2r-18r_Jr^2-12r^3}{2r_J^4}-\right.\\ &\qquad\left.\frac{6r^2(r+r_J)^2}{r_J^5}\ln\left(\frac{r}{r+r_J}\right)\right\}\end{aligned} \tag{A.19}$$

or, since $\langle v_r^2\rangle=P/\rho$,

$$\begin{aligned}\langle v_r^2\rangle &= \frac{G\mathcal{M}}{2r_J^4}\cdot\left\{r_J^3-2r_J^2r-18r_Jr^2-12r^3-\right.\\ &\qquad\left.\frac{12r^2(r+r_J)^2}{r_J}\ln\left(\frac{r}{r+r_J}\right)\right\}\end{aligned} \tag{A.20}$$

For circular orbits, $\langle v_r^2 \rangle = 0$, so $P = 0$, and the solution is even easier

$$\frac{dP}{dr} + \frac{(2P - Q)}{r} = -\frac{Q}{r} = -\rho \frac{d\Phi}{dr} = -\frac{G\mathcal{M}^2\, r_J}{4\pi} \cdot \frac{1}{r^3 (r + r_J)^3} \tag{A.21}$$

which implies

$$Q = \frac{G\mathcal{M}^2\, r_J}{4\pi r^2 (r + r_J)^3} \tag{A.22}$$

and

$$\langle v_\theta^2 \rangle = \langle v_\phi^2 \rangle = \frac{G\mathcal{M}}{r + r_J} \tag{A.23}$$

Note that this means that the circular velocity at any point is just

$$v_{\rm circ} = \left(\frac{G\mathcal{M}}{r + r_J} \right)^{1/2} \tag{A.24}$$

The law also implies a finite velocity dispersion at $r = 0$.

The only disadvantages of the Jaffe method is that the implied phase-space stellar distribution function is complicated (involving things called Dawson integrals), and the density and potential at $r = 0$ is infinite. However, even at $r = 0$, the total enclosed light of the Jaffe law is finite.

THE HERNQUIST LAW

[Hernquist 1990, *Ap.J.*, **356,** 359]

Today, the model that is most used to model the light distribution of an elliptical galaxy is the Herquist law

$$j(r) = \frac{\mathcal{L}}{2\pi} \cdot \frac{a}{r(r+a)^3} \tag{A.25}$$

It has many of the same advantages as the Jaffe model. It is a simple law that implies a finite total luminosity

$$\begin{aligned} \mathcal{L}_T &= \int_0^\infty 4\pi r^2 j(r)\, dr \\ &= \int_0^\infty \frac{2\mathcal{L}ar}{(r+a)^3}\, dr \\ &= 2\mathcal{L}a \left\{ \frac{a}{2(r+a)^2} - \frac{1}{r+a} \right\} \Bigg|_0^\infty = \mathcal{L} \end{aligned} \tag{A.26}$$

Like r_J, a is a scale factor: the space density that contains 1/4 of the total light. For comparison, the half-light space radius $r_{1/2} = (1+\sqrt{2})a$, and the effective radius is $R_e = 1.8153a$. Also, like the Jaffe law, the Hernquist model has a messy, but analytic form for the surface brightness

$$\begin{aligned} I(R) &= 2\int_R^\infty j(r) \cdot \frac{r}{(r^2-R^2)^{1/2}}\, dr \\ &= 2\int_0^\infty \frac{\mathcal{L}a}{2\pi} \cdot \frac{r}{r(r+a)^3(r^2-R^2)^{1/2}}\, dr \\ &= \frac{\mathcal{L}}{2\pi a^2(1-s^2)^2} \cdot \left\{(2+s^2)\chi(s) - 3\right\} \end{aligned} \tag{A.27}$$

where $s = R/a$, and

$$\begin{aligned} \chi(s) &= \frac{1}{\sqrt{1-s^2}} \quad \operatorname{sech}^{-1} s \quad \text{for} \quad 0 \leq s \leq 1 \\ &= \frac{1}{\sqrt{s^2-1}} \quad \sec^{-1} s \quad \text{for} \quad 1 \leq s \leq \infty \end{aligned} \tag{A.28}$$

If the mass-to-light ratio is constant, the Hernquist model's expression for the potential is simple and analytic. When the density profile

$$\rho(r) = \frac{\mathcal{M}}{2\pi} \cdot \frac{a}{r(r+a)^3} \tag{A.29}$$

is substituted into Poisson's equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\Phi}{dr}\right) = 4\pi G \cdot \frac{\mathcal{M}}{2\pi} \cdot \frac{1}{r(r+a)^3} \tag{A.30}$$

then

$$\begin{aligned} \frac{d\Phi}{dr} &= \frac{2G\mathcal{M}a}{r^2} \int \frac{r}{(r+a)^3}\, dr \\ &= \frac{2G\mathcal{M}a}{r^2} \left\{ -\frac{2r+a}{2(r+a)^2} + C \right\} \end{aligned} \tag{A.31}$$

Since the force must equal zero at $r = 0$,

$$C = \frac{2r+a}{2(r+a)^2} = \frac{1}{2a} \tag{A.32}$$

and

$$\begin{aligned} \Phi &= 2G\mathcal{M}a \int -\frac{2r+a}{2r^2(r+a)^2} + \frac{1}{2ar^2}\, dr \\ &= 2G\mathcal{M}a \int \frac{1}{2a(r+a)^2}\, dr \\ &= -\frac{G\mathcal{M}}{r+a} \end{aligned} \tag{A.33}$$

Also like the Jaffe law, the Hernquist law yields analytic expressions for the radial and tangential pressures in the case of isotropic and circular orbits.

$$\begin{aligned}\frac{dP}{dr} + \frac{(2P - Q)}{r} &= -\rho\,\frac{d\Phi}{dr} \\ &= \frac{\mathcal{M}}{2\pi} \cdot \frac{a}{r(r+a)^3} \cdot \frac{G\mathcal{M}}{(r+a)^2} \\ &= -\frac{G\mathcal{M}^2 a}{2\pi} \frac{1}{r(r+a)^5} \end{aligned} \tag{A.34}$$

For isotropic orbits where $Q = 2P$,

$$\begin{aligned} P &= -\frac{G\mathcal{M}^2 a}{2\pi} \int r^{-1}(r+a)^{-5} dr \\ &= \frac{G\mathcal{M}^2}{24\pi (r+a)^4} \cdot \left\{ \frac{12(r+a)^4}{a^4} \ln\left(\frac{r+a}{a}\right) - 25 \right. \\ &\qquad \left. - 52\left(\frac{r}{a}\right) - 42\left(\frac{r}{a}\right)^2 - 12\left(\frac{r}{a}\right)^3 \right\} \end{aligned} \tag{A.35}$$

and

$$\begin{aligned} \langle v_r^2 \rangle &= \frac{G\mathcal{M}}{12(r+a)} \left(\frac{r}{a}\right) \cdot \left\{ \frac{12(r+a)^4}{a^4} \ln\left(\frac{r+a}{a}\right) - 25 \right. \\ &\qquad \left. - 52\left(\frac{r}{a}\right) - 42\left(\frac{r}{a}\right)^2 - 12\left(\frac{r}{a}\right)^3 \right\} \end{aligned} \tag{A.36}$$

For circular orbits

$$Q = \frac{G\mathcal{M}^2 a}{2\pi (r+a)^5} \tag{A.37}$$

and

$$\langle v_\theta^2 \rangle = \langle v_{\mathrm{circ}}^2 \rangle = Q/\rho = \frac{G\mathcal{M} r}{(r+a)^2} \tag{A.38}$$

Finally, and perhaps most remarkably, in the case of isotropic and circular orbits, the expressions for σ_p^2, the *observed* velocity dispersion (which is the sum of the stellar motions all along the line-of-sight), is analytic. In general, this intensity-weighted measurement is given by

$$I(R)\sigma_p^2(R) = 2\int_R^\infty \left(1 - \beta\,\frac{R^2}{r^2}\right) P\,\frac{r}{\left(r^2 - R^2\right)^{1/2}}\,dr \tag{A.39}$$

where β describes the degree of orbital anisotropy

$$\beta = 1 - \frac{\langle v_\theta^2 \rangle}{\langle v_r^2 \rangle} \tag{A.40}$$

For isotropic orbits, $\beta = 0$, and after *a lot* of math,

$$\begin{aligned}\sigma_p^2(R) = \frac{G\mathcal{M}^2}{12\pi a^3 I(R)} \cdot \Bigg\{ &\frac{1}{2}\frac{1}{(1-s^2)^3} \cdot \\ &\Big[-3s^2\chi(s)\left(8s^6 - 28s^4 + 35s^2 - 20\right) \\ &-24s^6 + 68s^4 - 65s^2 + 6\Big] - 6\pi s \Bigg\}\end{aligned} \tag{A.41}$$

For circular orbits, $\beta = -\infty$, and

$$\sigma_p^2(R) = \frac{G\mathcal{M}^2 R^2}{2\pi a^5 I(R)} \cdot \left\{ \frac{1}{24(1-s^2)^4} \cdot \right.$$
$$\Big[-\chi(s)\left(24s^8 - 108s^6 + 189s^4 - 120s^2 + 120\right)$$
$$\left. - 24s^6 + 92s^4 - 117s^2 + 154 \Big] + \frac{\pi}{2s} \right\} \tag{A.42}$$

Like the Jaffe law, the Hernquist model has an analytic, though complicated expression for the stellar density in phase space, and possesses an infinite density at its center. The enclosed luminosity, however, is finite throughout.

$$\mathcal{L} = \int_0^r 4\pi r^2 \cdot \frac{\mathcal{L}}{2\pi} \cdot \frac{a}{r(r+a)^3}\, dr = \int_0^r \frac{2\mathcal{L}ar}{(r+a)^3}\, dr$$
$$= 2\mathcal{L}a \left\{ \frac{a}{2(r+a)^2} - \frac{1}{(r+a)} \right\} \tag{A.43}$$

Unlike the Jaffe law, the radial velocity dispersion at the center of a Hernquist model galaxy is $\langle v_r^2 \rangle = 0$. (But this is not necessarily a bad thing.)