Growth of Structure

At the time of the Big Bang, the universe was extremely smooth, but today it is extremely clumpy. How did we get to this state? Obviously the force of gravity acted over time to form the structures we see at $z = 0$. But exactly how did this occur? Did small objects form first, and collect gravitationally into larger structures, or did the universe begin with large clusters and superclusters. When did the formation of structure begin? Can we learn about the beginnings of structure from the microwave background? We can gain some insight into these questions by by examining the physical conditions of the early universe.

First, let's define two classes of perturbations. *Isothermal perturbations* are fluctuations that effect the matter of the universe, but not the surrounding radiation field. These are the types of fluctuations that occur today, since matter and radiation are decoupled. But in the early universe (in the era before recombination), this type of fluctuation may not have existed.

The second class of perturbations are called *adiabatic perturbations.* These effect both light and matter together. Such perturbations do not occur today, but they might have occurred at early times before decoupling.

Isothermal Perturbations

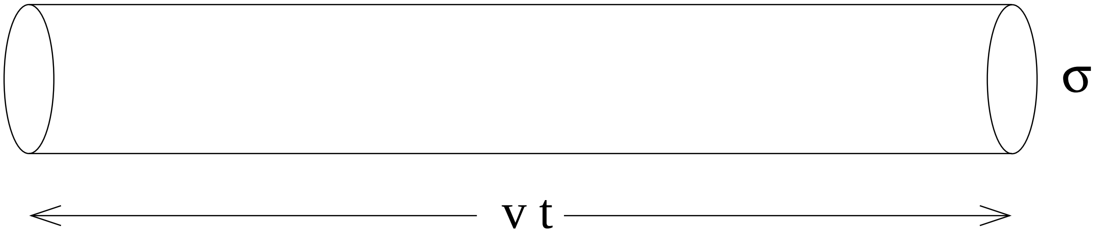
Fluctuations grow due to gravity; the gravitational force per unit mass is

$$F_{\rm grav} = \frac{G\mathcal{M}}{R^2} = \frac{4}{3}\pi G\rho R \qquad (4.01)$$

Now consider an isothermal perturbation, where gravity is attempting to move electrons (and protons) through a radition field. The velocity of the electrons is v; the cross section of the electrons to photon absorption is σ_T (the Thomson cross section). Because the electrons are moving through a medium filled with photons, they will experience a drag force (like a person running against the wind). Since the protons must follow the electrons, they will also (indirectly) feel this force. Per unit mass, the drag force is

$$F_{\rm drag} = \frac{aT^4(\sigma_T\, v)}{m_H c} \qquad (4.02)$$

(It's not difficult to see where this equation comes from: aT^4 is the energy density (ergs/cm^3) of the microwave background, and $\sigma_T\, v$ is the cylindrical volume swept out by the electron each second. So the numerator is the amount of energy encountered by the electron each second as it plows through the photons.)



If we compare $F_{\rm drag}$ to $F_{\rm grav}$, and explicitly put in the redshift dependence of T (3.11) and ρ (3.14) then

$$\frac{F_{\rm drag}}{F_{\rm grav}} = \frac{3\sigma_T\, vaT^4}{4\pi G m_H c R\,\rho} = \frac{3\sigma_T\, v\, aT_0^4(1+z)^4}{4\pi G m_H c R\,\rho_0(1+z)^3} \qquad (4.03)$$

Now consider R, the physical size of the perturbation being considered. In an Einstein de-Sitter universe, the time it takes an electron to move a distance R is approximately $v\,t$. If perturbations begin at the time of Big Bang, then by redshift z, the size is

$$R = \frac{2}{3}\frac{1}{H_0} v(1+z)^{-3/2} \tag{4.04}$$

If we use this to substitute for R, then

$$\frac{F_{\rm drag}}{F_{\rm grav}} = \frac{9\,\sigma_T a T_0^4 H_0}{8\pi G m_H c\,\rho_c\,\Omega_0}(1+z)^{5/2} \tag{4.05}$$

Numerically, this works out to be

$$\frac{F_{\rm drag}}{F_{\rm grav}} \sim 10^{-8}(1+z)^{5/2} \tag{4.06}$$

So note: before decoupling, when $z > 1500$, gravity cannot overcome the drag force of the microwave photons. So isothermal perturbations during this era do not grow.

Adiabatic Perturbations

If the radiation field is perturbed along with the matter, the excess energy contained in a region will diffuse outward and damp out the fluctuation. However, this damping does not occur instantaneously.

Let $\lambda = 1/(n_e \, \sigma_T)$ be the mean free path of a photon. In terms of λ, the time between scatterings for photons is λ/c, and, from statistics, the number of scatterings necessary for a photon to random walk a distance R is $(R/\lambda)^2$. For a fluctuation *not* to be damped out, the diffusion time must be longer than the age of the universe, *i.e.*,

$$\left(\frac{R}{\lambda}\right)^2 \left(\frac{\lambda}{c}\right) > \frac{2}{3}\frac{1}{H_0}(1+z)^{-3/2} \quad \Longrightarrow \quad R^2 > \frac{2}{3}\frac{c\,\lambda}{H_0}(1+z)^{-3/2} \tag{4.07}$$

Now if we substitute mass for radius and again note that the protons must follow the electrons, so that $\rho = n_e m_H$, then the condition for perturbation growth is

$$\mathcal{M} = \frac{4}{3}\pi\rho R^3 > \frac{4}{3}\pi\rho\left\{\frac{2c\, m_H}{3H_0\rho\,\sigma_T}(1+z)^{-3/2}\right\}^{3/2} \tag{4.08}$$

or, if we use (3.14) to substitute for density,

$$\mathcal{M} > \frac{4\pi}{3}\left\{\frac{2cm_H}{3H_0\,\sigma_T}\right\}^{-3/2}\left\{\frac{1}{\rho_c\Omega_0}\right\}^{1/2}(1+z)^{-15/4} \tag{4.09}$$

where ρ_c is the critical density of the universe. At decoupling, this works out to $\mathcal{M} \gtrsim 10^{13}\mathcal{M}_\odot$. This is somewhat of an underestimate, since λ is changing rapidly in the early universe. But it does demonstrate that adiabatic perturbations can only propagate if they are very large; small scale perturbations will be damped out.

The Jeans Mass

Another question we can ask concerns the minimum mass for gravitational collapse. For collapse to occur, the gravitational potential energy must overcome the thermal motion of the gas. If v_s is the sound speed,

$$\frac{G\mathcal{M}}{R} > \frac{1}{2} v_s^2 \tag{4.10}$$

If we substitute density for radius, then

$$2\,G\mathcal{M}\left(\frac{4\pi\rho}{3\mathcal{M}}\right)^{1/3} > v_s^2 \tag{4.11}$$

so

$$\mathcal{M}_J = \left(\frac{1}{8\pi}\right)^{\frac{3}{2}} G^{-\frac{3}{2}} \rho^{-\frac{1}{2}}\, v_s^3 \tag{4.12}$$

where $\mathcal{M}_J$ is called the Jeans mass. After de-coupling, matter will act as an ideal gas, so

$$v_s = \left(\frac{\gamma P}{\rho}\right)^{1/2} = \left(\frac{\gamma k T}{m_H}\right)^{1/2} \tag{4.13}$$

If we plug in the numbers for the universe shortly after decoupling *i.e.,* $z \sim 1000$, then $v_s \sim 4$ km s^{-1}, and the Jeans mass is $\mathcal{M}_J \sim 10^5 \mathcal{M}_\odot$. This is similar to the mass of a globular cluster.

In the early universe, however, the ideal gas law did not apply. At $z \sim 40,000$, radiation pressure dominated, and the energy density was much greater than the matter density. So

$$P = \frac{aT^4}{3} \quad \text{and} \quad \rho = \frac{aT^4}{c^2} \tag{4.14}$$

During this time, the sound speed was

$$v_s \sim \frac{c}{\sqrt{3}} \tag{4.15}$$

and the density was

$$\rho = aT_0^4(1+z)^4 \tag{4.16}$$

The Jeans mass was therefore $\mathcal{M}_J \sim 10^{16}\mathcal{M}_\odot$. Again this demonstrates that large fluctuations could grow in the early universe.

Growth of Perturbations

Can we connect the structure we see today to perturbations in the microwave background? The largest structures in today's universe are superclusters; these have overdensities of $\delta\rho/\rho \sim 2$. To see what that means, consider a region of slightly enhanced density, $\delta\rho$ within an Einstein de-Sitter universe. From (2.02), the equation for the expansion of this region is

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8}{3}\pi G(\rho + \delta\rho) = -\frac{kc^2}{R^2} \tag{4.17}$$

while the equation of the expansion of the rest of the universe is

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8}{3}\pi G\rho = 0 \tag{4.18}$$

If $\delta\rho$ is small, then the de-celerations of these two regions are similar. To first order, we can therefore subtract these two equations to get

$$\frac{\delta\rho}{\rho} = \frac{3kc^2}{8\pi G\rho R^2} \tag{4.19}$$

Since the density of the universe, $\rho \propto R^{-3}$, this means that

$$\frac{\delta\rho}{\rho} = \frac{3kc^2}{8\pi G\mathcal{M}} R \tag{4.20}$$

or

$$\frac{\delta\rho}{\rho} \propto R \tag{4.21}$$

In other words, the amplitude of a small density perturbation will grow linearly with the size of the universe due to the Hubble expansion alone, *i.e.*, without any assistance from gravity. This puts a limit on the amplitude of the initial density perturbations: for example, if a supercluster now has $\delta\rho/\rho \sim 2$, at decoupling, the initial density fluctuation must have been at least 1000 times smaller.

The Sachs-Wolfe Relation

We can observe the density fluctuations of the universe directly in the microwave background via the Sachs-Wolfe Relation. Consider a density contrast $\delta_x = \delta\rho/\rho$ over some region of the universe, which, in co-moving coordinates is u. Because the universe is expanding, the physical size of this region will expand with time, so that the actual size of the region is Ru. Now consider the gravitational potential of the region. From simple Newtonian physics, this is

$$\phi = \frac{G\mathcal{M}}{Ru} = \frac{4\pi G R^3 u^3 \delta_x \rho}{3Ru} = \frac{4}{3}\pi G R^2 u^2 \delta_x \rho \tag{4.22}$$

Next, let's substitute for the mean density of the universe. If we use the definition of Ω (1.13), then

$$\rho = \Omega \rho_c = \frac{3}{8\pi G} H^2 \Omega \tag{4.23}$$

and

$$\phi = \frac{1}{2}\Omega\, R^2 u^2 H^2 \delta_x \tag{4.24}$$

Now let's examine this equation. If we make the (good) approximation that the density of the universe at high-z was very near critical (*i.e.,* the Einstein-de Sitter equations hold), then from (1.24) and (1.28)

$$R \propto t^{2/3} \quad \text{and} \quad H \propto \frac{1}{t} \quad \Longrightarrow \quad H \propto R^{-3/2} \tag{4.25}$$

Since from (4.21), $\delta_x \propto R$

$$\phi \propto \Omega R^2 R^{-3} R \quad \Longrightarrow \quad \phi = \text{Constant} \tag{4.26}$$

In other words, although the fluctuation grows with time, the potential due to the fluctuation does not. We can therefore substitute into (4.24) the current values of the universe

$$\phi = \frac{1}{2}\Omega_0 H_0^2 (R_0 u)^2 \delta_x(0) \tag{4.27}$$

Now a photon passing through this potential will experience a gravitational redshift of the order of

$$\frac{\delta\lambda}{\lambda} = \frac{\phi}{c^2} \tag{4.28}$$

Thus, the microwave photons of the region will appear cooler by

$$\frac{\delta T}{T} = \frac{\Omega_0 H_0^2 (R_0 u)^2 \delta_x(0)}{2c^2} \tag{4.29}$$

Let's put in some typical numbers. Superclusters have typical sizes of $Ru \sim 100$ Mpc and density contrasts of $\delta_x \sim 2$. For a Hubble Constant of $H_0 = 75$ km s^{-1} Mpc^{-1}, this means that $\delta T/T \sim 6 \times 10^{-4}$ Mpc. This is not too far from the $\sim 3 \times 10^{-5}$ value that is observed. (Note that this effect is most easily observable at large scales: at small scales, the gravitational redshift is much harder to detect (and can be lost in the noise of other sources.)