Einstein's Universe

Newtonian gravity gave the energy of the universe as

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8}{3}\pi G\rho_0\left(\frac{R_0}{R}\right)^3 = \frac{2E}{R^2} \tag{2.01}$$

In the relativisitic case, the equation is very similar

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8}{3}\pi G\rho_0\left(\frac{R_0}{R}\right)^3 = -\frac{kc^2}{R^2} + \frac{\Lambda}{3} \tag{2.02}$$

Here, k is constant which defines the curvature of the universe. Note that k only appears when divided by R^2 (which is not measureable). Thus, to simplify things, we can choose the units of R such that k is either 0, 1, or -1. The value $k = 1$ represents a universe with positive curvature, like a sphere; $k = -1$ reflects negative curvature, like a saddle. In the critical case, where $k = 0$, the universe is flat.

The other new variable Λ is a Cosmological Constant. You can think of it as a pressure term which supplies a new repulsive (or attractive) force that is directly proportional to distance,

$$\ddot{R} = \Lambda R \tag{2.03}$$

If one wants the universe to be static, then one needs to include this additional term to counter-act the attractive force of gravity. (There are other reasons to add in a Cosmological Constant, but this was the original justification.)

In a relativistic universe, you must also take into account that the distance between two co-moving points is not simply du. In normal space, the separation between two points is

$$ds^2 = du^2 = dx^2 + dy^2 + dz^2 \tag{2.04}$$

The total distance, s is then found by integrating along the path. In space-time, the interval between these two points is

$$ds^2 = c^2 dt^2 - du^2 \tag{2.05}$$

where now, the path in both time and space must be integrated. Finally, in the case of an expanding universe, the space distance between two points is $R\,du$, where R is the size of the universe at the time of the measurement. So the distance between two points is

$$ds^2 = c^2 dt^2 - R^2 du^2 \tag{2.06}$$

This is called the Robertson-Walker metric. Note that for light (which travels at the speed of light), $ds = 0$.

An additional complication comes from the fact that space-time is not necessarily flat. In plane geometry, the Cartesian coordinates are

$$du^2 = dx^2 + dy^2 + dz^2 \tag{2.07}$$

In spherical coordinates, this can be written as

$$du^2 = d\xi^2 + \xi^2 d\theta^2 + \xi^2 \sin^2\theta \, d\phi^2 \tag{2.08}$$

In curved space, however, the equation is slightly different:

$$du^2 = \frac{d\xi^2}{1 - k\xi^2} + \xi^2 d\theta^2 + \xi^2 \sin^2\theta \, d\phi^2 \tag{2.09}$$

If $k = 0$, the equation reduces to flat-space spherical coordinates.

Cosmological Luminosity Distance

Let's put this all together and compute how the flux from a source varies with distance. Locally, we know that $F \propto 1/r^2$, but in an expanding universe, this will not be the case.

First, consider that the flux of n photons per second emitted from a source at redshift z can be expressed as

$$F = \frac{n \cdot h\nu_e}{dt_e} = \frac{\text{photons} \cdot \text{energy}}{\text{time}} \tag{2.10}$$

while the flux observed is

$$f = \frac{n \cdot h\nu_0}{dt_0} \cdot \frac{1}{d^2} \tag{2.11}$$

So, we need to relate ν_0, dt_0, and d to ν_e, dt_e, and z. First, from the definition of redshift

$$\nu_0 = \nu_e/(1+z) \tag{1.34}$$

Next, note that $dt_0 \neq dt_e$: because the source is moving, time dilation occurs, causing us to measure the source's clocks to be moving slower,

$$t_0 = \frac{t_e}{\left\{1 - (v/c)^2\right\}^{1/2}} \tag{2.12}$$

In addition, the interval between two pulses will appear longer, because the distance between us and the source is every increasing. (A second pulse has longer distance to travel.) This extra time is

$$\Delta t_0 = \frac{t_e(v/c)}{\left\{1 - (v/c)^2\right\}^{1/2}} \tag{2.13}$$

Thus,

$$t_0 + \Delta t_0 = t_e \frac{1 + (v/c)}{\left\{1 - (v/c)^2\right\}^{1/2}} = t_e \left\{ \frac{1 + (v/c)}{1 - (v/c)} \right\}^{1/2} = (1 + z) \tag{2.14}$$

or

$$dt_0 = (1 + z) dt_e \tag{2.15}$$

So, with these two terms, we have

$$f = \frac{nh\nu_0}{dt_0} \frac{1}{d^2} = \frac{nh\nu_e}{dt_e} \frac{1}{(1+z)^2} \frac{1}{d^2} = \frac{F}{(1+z)^2 d^2} \tag{2.16}$$

Now we have to determine the "proper distance," d. Let's start with the Robertson-Walker metric

$$ds^2 = c^2 dt^2 - R\, du^2 \tag{2.06}$$

which, for light implies

$$c^2 dt^2 = R^2 du^2 = R^2 \left\{ \frac{d\xi^2}{1 - k\xi^2} + \xi^2 d\theta^2 + \xi^2 \sin^2\theta\, d\phi^2 \right\} \tag{2.17}$$

Since any point in the universe is as good as any other point, we will place ourselves at the coordinate system's origin. That makes the light's path purely radial and simplifies things enormously, since $d\theta = 0$ and $d\phi = 0$. So (2.17) is just

$$c^2 dt^2 = R^2 \left\{ \frac{d\xi^2}{1 - k\xi^2} \right\} \tag{2.18}$$

The proper distance $d_p = \int R du$ can now be found by taking the square root of both sides of the equation, and integrating over the light path, starting at the origin at the universe's time t_1, and going to us (at coordinate u) at time t_0.

$$\int_{t_0}^{t_1} \frac{c}{R} dt = \int_u^0 \frac{d\xi}{(1 - k\xi^2)^{1/2}} \tag{2.19}$$

For a flat ($k = 0$) universe, the right-hand integral is trivial, and, thanks to equation (1.23) $R = at^{2/3}$, the left-hand integral is only slightly harder. Therefore, (2.19) reduces to

$$\frac{3c}{a} t_0^{1/3} - \frac{3c}{a} t_1^{1/3} = u \tag{2.20}$$

Now recall that the definition of redshift (1.35)

$$(1 + z) = \frac{R_0}{R_1} = \frac{at_0^{2/3}}{at_1^{2/3}} = \left(\frac{t_0}{t_1}\right)^{2/3} \tag{2.21}$$

With this substitution, (2.20) becomes

$$u = \frac{3c}{a} t_0^{1/3} \left\{1 - (1 + z)^{-1/2}\right\} \tag{2.22}$$

Finally, note that $\dot{R} = \frac{2}{3} at^{-1/3}$, so

$$u = \frac{2c}{\dot{R}_0} \left\{1 - (1 + z)^{-1/2}\right\} \tag{2.23}$$

and, through the definition of $H_0 = \dot{R}_0/R_0$,

$$d_p = R_0 u = \frac{2c}{H_0} \left\{1 - (1 + z)^{-1/2}\right\} \tag{2.24}$$

This equation, coupled with (2.16) gives us the relation between the emitted flux F, and the observed flux, f, in a flat universe.

$$f = \frac{F}{(1+z)^2} \cdot \frac{H_0^2}{4c^2 \left\{1 - (1+z)^{-1/2}\right\}^2} \tag{2.25}$$

Note that this equation is not a simple inverse square law. In an expanding universe, even if you know the emitted flux of an object, you cannot determine the object's distance without knowing something about the cosmology. Alternatively, if one desires to express an object's observed flux as an inverse square law, one does so as

$$f = \frac{F}{d_L^2} \tag{2.26}$$

where d_L is called the *luminosity distance.* Obviously, from (2.25), luminosity distance differs from proper distance by

$$d_L = d_p(1+z) \tag{2.27}$$

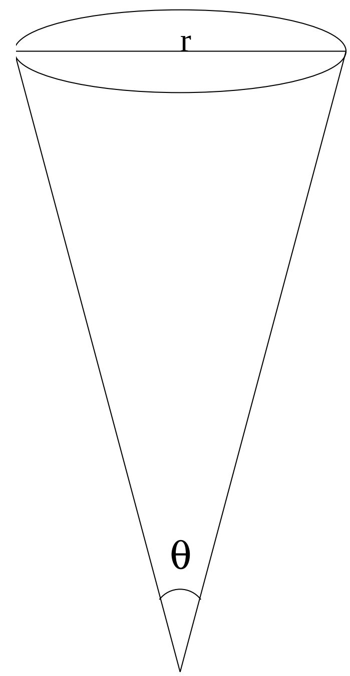
In fact, the equation for proper distance (2.24) derived above is only good for an Einstein-de Sitter universe. In the general case, where $q_0 \neq 1/2$, the math is more tedious, though not extraordinarily difficult. For the general case of a $\Lambda = 0$ universe, the equation for proper distance is

$$d_p = \frac{c}{H_0 q_0^2 (1+z)} \left\{ q_0 z + (q_0 - 1)\left[(2q_0 z + 1)^{1/2} - 1\right]\right\} \tag{2.28}$$

Angular Diameter Distance

Astronomers have two ways of measuring distance: through the use of a "standard candle," where you know (or think you know) the intrinsic flux from a source, and through the use of a "standard yardstick," where you know the physical size of an object.

Consider a standard galaxy with linear size r at redshift z. Under normal Euclidean geometry, the angular size the galaxy subtends would be inversely proportional to distance. However, in an expanding relativistic universe, the calculation is a bit more complicated.



Again, let's begin with the Robertson-Walker metric

$$\begin{aligned} ds^2 &= c^2 dt^2 - R^2 du^2 \\ &= c^2 dt^2 - R^2 \left\{ \frac{d\xi^2}{1 - k\xi^2} + \xi^2 d\theta^2 + \xi^2 \sin^2\theta d\phi^2 \right\} \end{aligned} \quad (2.29)$$

If the galaxy is in the plane of the sky, then the radial distance to both sides of the galaxy is the same. Thus, $d\xi = 0$. Similarly, since both sides of the galaxy are being observed at the same time, $dt = 0$, and the separation of the two sides of the galaxy is simply $\int ds = s$. Finally, we will choose our coordinate system such that the angle subtended by the galaxy is entirely in the θ direction, so that $d\phi = 0$.

Thus from (2.29)

$$ds^2 = -R^2 du^2 = -R^2 \xi^2 d\theta^2 \tag{2.30}$$

If we integrate this equation, and choose a coordinate system so that θ is positive, then

$$\theta = \frac{s}{R\xi} \tag{2.31}$$

Let's get rid of the R (the size of the universe at the redshift of the galaxy) and substitute R_0 (the size of the universe today), using (1.35), $(1 + z) = R_0/R$. Then

$$\theta = \frac{s}{R\xi} = \frac{s(1+z)}{R_0 \xi} \tag{2.32}$$

Now recall that $R_0\xi = R_0 u$ is the "proper distance" to the galaxy (2.24) [or (2.28) for the general case]. So, for a flat universe,

$$\theta = \frac{H_0 s(1+z)}{2c\left\{1 - (1+z)^{-1/2}\right\}} \tag{2.33}$$

Again, this is different from the results of Euclidean geometry, where $\theta = s/\xi$. Like luminosity, we can define an object's *angular size distance* as the distance needed to reproduce a Euclidean result, *i.e.,*

$$\theta = \frac{s}{d_A} \tag{2.34}$$

where

$$d_A = d_p\,(1+z)^{-1} = d_L(1+z)^{-2} \tag{2.35}$$

Equation (2.33) has a very interesting property. According to the equation, the angular size subtended by a galaxy is proportional to s/R, *i.e.,* the size of the galaxy over the size of the universe. At large redshift, when the universe was small, our "standard yardstick" extended over a larger fraction of the universe than it does today. Consequently, when viewed in today's universe, the object appears larger.

Evolution in the Universe

A large amount of work in cosmology centers around defining two numbers – the Hubble Constant, H_0, and the deceleration parameter, q_0. As we have seen, two ways to do this are through measurements of luminosity and angular size. The behavior of $m(z)$, the apparent magnitude of a standard candle as a function of redshift, and $\theta(z)$, the angular size of a standard yardstick as a function of redshift, both depend sensitively on q_0 and H_0. (This is especially true at redshifts $z > 1$.) So, in theory, the measurement of these two quantities should be straightforward. In practice, of course, there are many difficulties involved in measurement of q_0, not the least of which is the question of evolution. In general, there are two types of evolution which astronomers frequently encounter. The first is *luminosity evolution*, *i.e.,* how the brightness (or size) of an individual object changes with time. Since we cannot observe this evolution in real time, luminosity evolution must be inferred indirectly from the observations. This is easier said than done, since in addition to luminosity evolution, there may also be *density evolution,* in which the number of objects changes with time. Often times, there is no easy way to disentangle these two effects.

Volume Element of the Universe

To interpret surveys of objects in the context of evolution, astronomers often need to know the volume of the surveyed region. Hence the question, "what is the volume of the universe between two redshifts?" In general, the volume element of anything in spherical coordinates is of the form

$$dV = r^2 \sin\theta \, dr \, d\theta \, d\phi \tag{2.36}$$

The angle variables θ and ϕ are unaffected by cosmology and act the same as they would in Euclidean space. That leaves r and dr. Since the shell is concentric with the earth (*i.e.,* both ends of the shell are being observed simultaneously, so $dt = 0$), r is the angular size distance, d_A However, dt is not zero when one considers the thickness of the shell: for that, you need to consider proper distance, Rdu at the redshift being considered. In addition, proper distance, in itself, is not very useful – the observable variable is z. Thus the volume element of the universe is

$$dV = r_A^2 \sin\theta \frac{dr_p}{dz} dz d\theta d\phi \tag{2.37}$$

This is not difficult to evaluate: in the general case, the volume element is

$$dV = r_A^2 \frac{c}{H_0 \left(2q_0 z + 1\right)^{1/2} (1+z)^2} \sin\theta \, dz \, d\theta \, d\phi \tag{2.38}$$