Principles of Cosmology

The basic assumptions of Cosmology are

1) The Universe is homogeneous – every observer sees the same thing.

2) The Universe is isotropic – there is no preferred direction in the universe.

The implication of these two statements is that the universe must either be static, or have purely radial motion. (For example, if the universe rotated, then the preferred axis of rotation would violate the assumption of isotropy.)

Let's assume that the universe is dynamic, and let

$\vec{u} =$ the co-moving coordinates of an object
$R(t) =$ the motion (expansion or contraction) of the universe
$\ell =$ the measured distance to an object

From these definition, the observed velocity of a galaxy is

$$v = \frac{d\ell}{dt} = \frac{d}{dt}(Ru) = \dot{R}u \tag{1.01}$$

Now, neither R nor u is observable, but ℓ is. So, let's substitute ℓ for u

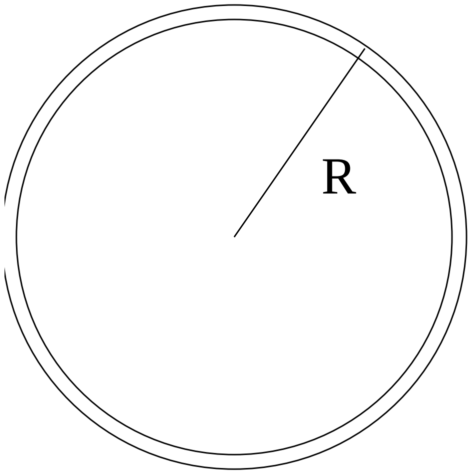
$$v = \dot{R}\left(\frac{\ell}{R}\right) = \left(\frac{\dot{R}}{R}\right)\ell \tag{1.02}$$

Now, let's define the variable $H(t) = \dot{R}/R$, with the units of inverse time. (The unobservable distance unit attached to R has cancelled out!) $H(t)$ is the Hubble parameter; its value today, $H_0 = \dot{R}_0/R_0$ is the Hubble Constant. $H(t)$ is the fractional rate of expansion of the universe.

The Newtonian Universe

Consider a Newtonian universe, where the Newtonian laws of gravity apply. Let's pick a center to the universe, and consider the motion of a shell of material at distance R from the center. (R can be the extreme limit of the universe, if you wish, or can be just a small region.)

Recall from freshman physics that, if the universe is homogeneous and isotropic, the matter outside the shell will have no gravitational effect on its motion; the deceleration of the shell only depends on the matter interior to it.



So the deceleration of the shell is

$$a = \ddot{R} = -\frac{G\mathcal{M}}{R^2} \tag{1.03}$$

If we multiply each side by $\dot{R}$ and integrate over time, we get

$$\int \dot{R}\,\ddot{R}\,dt = -\int \frac{G\mathcal{M}}{R^2}\dot{R}\,dt \tag{1.04}$$

Since $\ddot{R}$ is the derivative of $\dot{R}$, and $\dot{R}$ is the derivative of R, both integrals are easy (of the form $\int u\,du$). The result is

$$\frac{1}{2}\dot{R}^2 - \frac{G\mathcal{M}}{R} = E \tag{1.05}$$

where E, the total energy, is the constant of integration. Note that this is nothing but energy conservation! If the potential energy of the universe is greater than its kinetic energy, $E < 0$, and eventually there will be a collapse. If the kinetic energy is greater than the potential energy, $E > 0$.

Now let's parameterize how fast the universe is decelerating. Again, since we can't measure the size of the universe (and, in fact, the size is irrelevant, since the equations hold for small sections of the universe as well), we must somehow make the units of R disappear. If we take the energy conservation equation, multiply it by two, and divide by R^2, then

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{2G\mathcal{M}}{R^3} = \frac{2E}{R^2} \tag{1.06}$$

or, since $\ddot{R} = -G\mathcal{M}/R^2$,

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{2\ddot{R}}{R} = \frac{2E}{R^2} \tag{1.07}$$

If we multiply and divide the second term in the equation by $(R/\dot{R})^2$, then

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{2\ddot{R}}{R}\left(\frac{R}{\dot{R}}\right)^2\left(\frac{\dot{R}}{R}\right)^2 = \frac{2E}{R^2} \tag{1.08}$$

$$\left(\frac{\dot{R}}{R}\right)^2 + 2\left(\frac{\ddot{R}R}{\dot{R}^2}\right)\left(\frac{\dot{R}}{R}\right)^2 = \frac{2E}{R^2} \tag{1.09}$$

We can now define a dimensionless deceleration parameter

$$q(t) = -\frac{\ddot{R}R}{\dot{R}^2} \tag{1.10}$$

Equation (1.09) is then

$$H^2 - 2H^2q = \frac{2E}{R^2} \tag{1.11}$$

or

$$(1 - 2q) = \frac{2E}{H^2R^2} \tag{1.12}$$

From this definition, it is clear that if $q < 1/2$, $E > 0$, the universe is unbound. If $q > 1/2$, $E < 0$, and we have a bound universe. $q = 1/2$ implies $E = 0$, which is a critical universe.

We can parameterize the universe in yet another way. If ρ_c is the critical density of the universe, then the cosmological density parameter

$$\Omega(t) = \rho/\rho_c \tag{1.13}$$

Obviously, if the universe is bound, $\Omega > 1$, but if it's unbound, $\Omega < 1$.

The Age of the Universe

What is the age of the universe as a function of H_0 and q_0? To compute this number, we can start with energy conservation

$$\frac{1}{2}\dot{R}^2 - \frac{G\mathcal{M}}{R} = E \tag{1.05}$$

and as before, multiply through by $2/R^2$

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{2G\mathcal{M}}{R^3} = \frac{2E}{R^2} \tag{1.06}$$

The mass of the universe (which is presumed to be constant) can be evaluated using the universe's present size and density, *i.e.*,

$$\mathcal{M} = \frac{4}{3}\pi R_0^3\,\rho_0 \tag{1.14}$$

So

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8}{3}\pi G\,\rho_0\left(\frac{R_0}{R}\right)^3 = \frac{2E}{R^2} \tag{1.15}$$

First, consider an empty (Milne) universe. In this case, $\rho_0 = 0$, so equation (1.15) becomes

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{2E}{R^2} \tag{1.16}$$

or

$$\dot{R} = (2E)^{1/2} \tag{1.17}$$

This is easily integrated from $t = 0$ to the present to get

$$R(t) = (2E)^{1/2}\,t \tag{1.18}$$

The age of the universe is therefore

$$t = \frac{R}{(2E)^{1/2}} \tag{1.19}$$

This can't be evaluated, but we can substitute $\dot{R}$ for $(2E)^{1/2}$ (using 1.17). So

$$t = \frac{R}{\dot{R}} = \frac{1}{H} \tag{1.20}$$

So, for a Milne universe, the present age is $t_0 = 1/H_0$.

For an Einstein-de Sitter (critical) universe, it's a bit more complicated. In this case, $E = 0$, and equation (1.15) becomes

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8}{3}\pi G\rho_0 \left(\frac{R_0}{R}\right)^3 = 0 \tag{1.21}$$

If we multiply through by R^2, then the equation becomes

$$\dot{R}^2 - \frac{8}{3}\pi G\rho_0 R_0^3\, R^{-1} = 0 \tag{1.22}$$

or

$$\dot{R} = \left\{\frac{8}{3}\pi G\rho_0 R_0^3\right\}^{1/2} R^{-1/2} \tag{1.23}$$

This is a differential equation, and solving differential equations can be a pain. Let's guess that the solution is

$$R = at^{2/3} \tag{1.24}$$

where a is some constant. Does this work? Let's check.

$$\dot{R} = \left\{ \frac{8}{3} \pi G \rho_0 R_0^3 \right\}^{1/2} R^{-1/2} \tag{1.23}$$

$$\dot{R} = \frac{2}{3} a t^{-1/3} = \left\{ \frac{8}{3} \pi G \rho_0 R_0 \right\}^{1/2} a^{-1/2} t^{-1/3} \tag{1.25}$$

Thus, the guess works if

$$a^{3/2} = \frac{3}{2} \left\{ \frac{8}{3} \pi G \rho_0 R_0^3 \right\}^{1/2} \tag{1.26}$$

Now, to calculate the age of the universe, start with $R = at^{2/3}$ and $\dot{R} = (2/3)at^{-1/3}$. We can again get rid of the unobservables by taking a ratio

$$\left(\frac{\dot{R}}{R} \right) = \frac{(2/3)at^{-1/3}}{at^{2/3}} = \frac{2}{3} \frac{1}{t} \tag{1.27}$$

So

$$t = \frac{2}{3} \left(\frac{R}{\dot{R}} \right) = \frac{2}{3} \frac{1}{H} \tag{1.28}$$

and, at the present time

$$t_0 = \frac{2}{3} \frac{1}{H_0} \tag{1.29}$$

The case where $E \neq 0$ also has a solution, but the form of the solution is much more complicated. You can't write the solution as $R(t)$; you have to parameterize the equations through a third variable, θ, and write $R(\theta)$ and $t(\theta)$. The math is not difficult, but it is sufficiently tedious that we won't go into it here.

What effect does the universal expansion have on light? To see this, first consider a particle with velocity $\vec{v}$ moving past an observer at point 1 on its way to an observer at point 2. By the time it gets there, the universe has expanded; specifically, if the particle has traveled $v(t)dt$, so the expansion velocity of the universe is $H\, v(t)dt$. Consequently, the observer will measure the particle's velocity to be

$$v(t+dt) = v(t) - \left(\frac{\dot{R}}{R}\right) v(t)dt \tag{1.30}$$

A bit of algebra yields

$$\frac{v(t+dt) - v(t)}{v(t)} = -\left(\frac{\dot{R}}{R}\right) dt \tag{1.31}$$

or

$$\frac{dv}{v} = -\frac{dR}{R} \tag{1.32}$$

which simply integrates to $v = R^{-1}$. The same argument applies to the frequency of a photon, yielding

$$\nu(t) = R^{-1} \tag{1.33}$$

If we now define redshift as

$$(1+z) = \frac{\nu_e}{\nu_{obs}} = \frac{\lambda_{obs}}{\lambda_e} \tag{1.34}$$

then

$$(1+z) = \frac{R_0}{R(t)} \tag{1.35}$$

One additional point should be made here. Recall that the equation for the expansion of the universe was derived by assuming that the motion of a shell of matter was affected by only one force – the force of gravity. However, in theory, other forces could influence the shell's motion. For example, if there was a pressure term on the shell, then instead of

$$\ddot{R} = -\frac{G\mathcal{M}}{R^2} = -\frac{4}{3}\pi G\rho R \tag{1.36}$$

the acceleration on the shell would be

$$\ddot{R} = -\frac{4}{3}\pi G(\rho + 3p)R \tag{1.37}$$

The pressure term can then be propagated through and produce an extra term on the right side of the equation of motion.