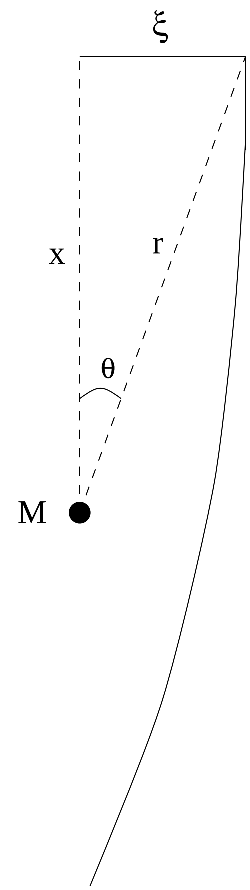
Gravitational Lenses

Consider a photon moving past a point of mass, $\mathcal{M}$, with an starting "impact parameter," ξ. From classical Newtonian gravity, the photon will undergo an acceleration perpendicular to the direction of its motion. Under the Born approximation, the amount of this deflection can be calculated simply by integration along the path.



$$\frac{dv_\perp}{dt} = \frac{G\mathcal{M}}{r^2}\sin\theta \tag{11.01}$$

If we substitute $dx = c\,dt$, then

$$\begin{aligned} v_\perp &= \frac{G\mathcal{M}}{c}\int_{-\infty}^{\infty}\frac{1}{x^2+\xi^2}\cdot\frac{\xi}{(x^2+\xi^2)^{1/2}}\,dx \\ &= \frac{G\mathcal{M}\xi}{c}\int_{-\infty}^{\infty}\left(x^2+\xi^2\right)^{-3/2}\,dx \quad (11.02) \end{aligned}$$

The integral in (11.02) is analytic, and works out to $2/\xi^2$. So

$$v_\perp = \frac{2G\mathcal{M}}{\xi c} \tag{11.03}$$

and the Newtonian deflection angle (in the limit of a small deflection) is

$$\alpha = \frac{v_\perp}{c} = \frac{2G\mathcal{M}}{\xi c^2} \tag{11.04}$$

In the general relativistic case, however, gravity affects both the spatial and time component of photon's path, so the actual bending is twice this value. Thus, we define the angle of deflection, otherwise known as the Einstein angle, as

$$\alpha = \frac{4G\mathcal{M}}{\xi c^2} \tag{11.05}$$

To understand the geometry of a gravitational lens, let's first define a few terms. Let

D_d	=	distance from the observer to the lens
D_s	=	distance from the observer to the light source
D_{ds}	=	distance from the lens to the source
$\vec{\beta}$	=	true angle between the lens and the source
$\vec{\theta}$	=	observed angle between the lens and the source.
$\vec{\xi}$	=	distance from the lens to a passing light ray
$\vec{\alpha}$	=	the Einstein angle of deflection

Note that β, θ, α, and ξ are all vectors, and calculations must deal with negative, as well as positive angles. For a simple point-source lens, the vectorial components of the angles make no difference, but in systems where the lens is complex (*i.e.*, several galaxies/clusters along the line-of-sight) the deflection angles must be added vectorially.

In practice, α, β, and θ are all very small, so small angle approximations can be used. Also, D_d, D_s, and D_{ds} are all much bigger than ξ. Thus, the deflection can be considered instantaneous (*i.e.*, we can use a "thin lens" approximation).

The geometry of a simple gravitational lens system is laid out in the figure. From $\triangle OSI$ and the law of sines,

$$\frac{\sin(180-\alpha)}{D_s} = \frac{\sin(\theta-\beta)}{D_{ds}} \tag{11.06}$$

Since all the angles are small, $\sin(\theta-\beta) \approx \theta-\beta$, and $\sin(180-\alpha) = \sin\alpha \approx \alpha$. So

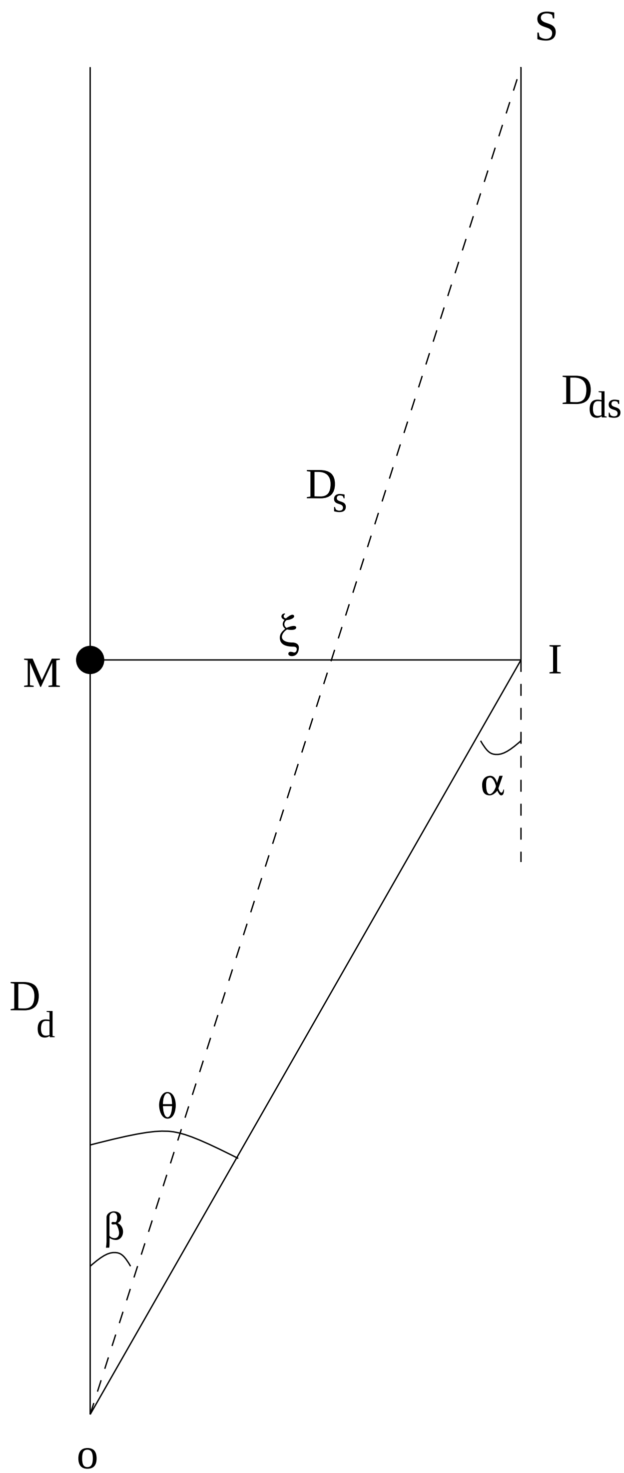
$$\vec{\beta} = \vec{\theta} - \frac{D_{ds}}{D_s}\vec{\alpha} \tag{11.07}$$

(Again, for single point lenses, the fact that the angles are vectors are irrelevant. For simplicity, I will therefore drop the vector signs for the rest of these derivations.)

Since we have already seen that

$$\alpha = \frac{4G\mathcal{M}}{\xi c^2} \tag{11.05}$$

and, from the simple geometry of small angles, $\theta = \xi/D_d$,



$$\beta = \theta - \left(\frac{4G\mathcal{M}}{c^2} \frac{D_{ds}}{D_d D_s} \right) \cdot \frac{1}{\theta} \tag{11.08}$$

In other words, we have a relation between β and θ. But note: for a given value of β, there is more than one value of θ that will satisfy the equation. This is a general theorum of lenses. For non-transparent lenses (such as the Schwarzschild lens being considered) there will always be an even number of images; for transparent lenses, there is always an odd number of images.

To simplify, let's define the characteristic bending angle, α_0, as a quantity that depends only on the mass of the lens and the distances involved

$$\alpha_0 = \left(\frac{4G\mathcal{M}}{c^2} \frac{D_{ds}}{D_d D_s} \right)^{1/2} \tag{11.09}$$

so that

$$\beta = \theta - \frac{\alpha_0^2}{\theta} \tag{11.10}$$

If β and θ are co-planar (as they will be for a point source lens), then (11.10) is equivalent to the quadratic equation

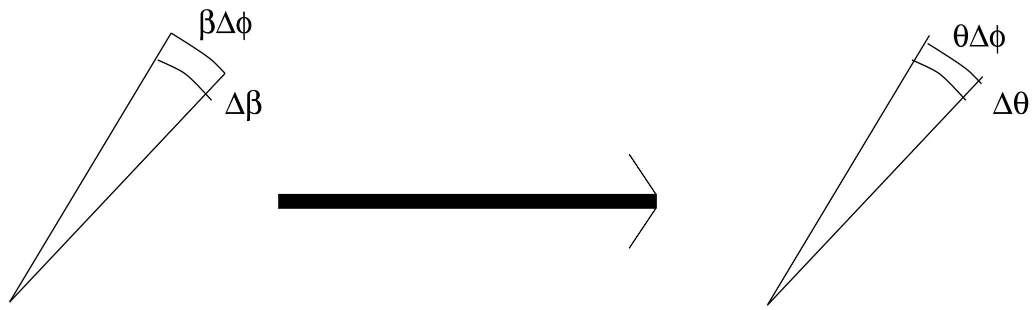
$$\theta^2 - \beta\theta - \alpha_0^2 = 0 \tag{11.11}$$

whose solution is

$$\theta = \frac{1}{2} \left(\beta \pm \sqrt{4\alpha_0^2 + \beta^2} \right) \tag{11.12}$$

This defines the location of the gravitional lens images as a function of the true position of the source with respect to the lens, β, and α_0.

Magnification for a Schwarzschild Lens



The relation

$$\theta = \frac{1}{2}\left(\beta \pm \sqrt{4\alpha_0^2 + \beta^2}\right) \tag{11.12}$$

implies that at least one image will be magnified. To see this, let's lay out a polar coordinate system with the lens at the center, and consider the light passing through a differential area, dA. At the lens, this area element is $dA = \beta\Delta\phi\,\Delta\beta$. However, due to the gravitational lens, the angles are distorted, so that the observer gets sees this light squeezed into area $dA' = \theta\Delta\phi\,\Delta\theta$. Thus, the lens has "focussed" the light from area dA to area dA'. The magnification will therefore be the ratio of the two areas

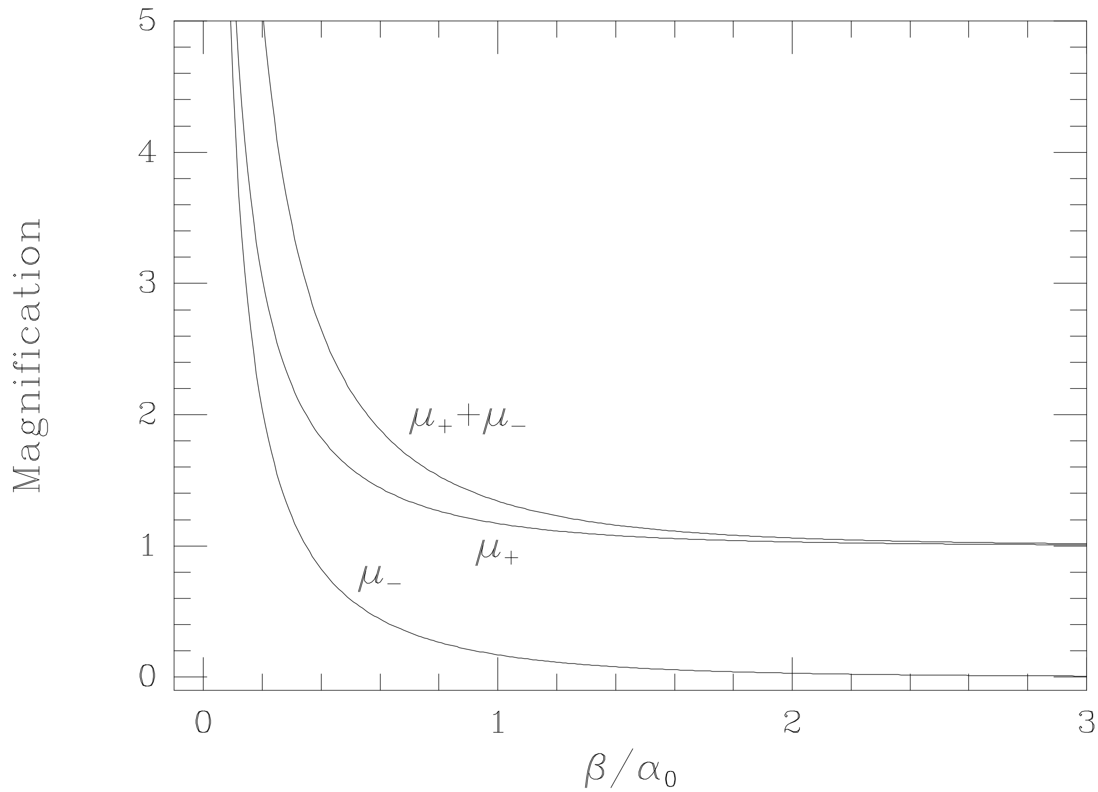
$$\mu = \left|\frac{\theta\Delta\phi\,\Delta\theta}{\beta\Delta\phi\Delta\beta}\right| = \left|\frac{\theta}{\beta}\frac{\Delta\theta}{\Delta\beta}\right| = \left|\frac{\theta}{\beta}\frac{d\theta}{d\beta}\right| \tag{11.13}$$

The last term is just the derivative of (11.12), or

$$\frac{d\theta}{d\beta} = \frac{1}{2}\left(1 \pm \frac{\beta}{\sqrt{4\alpha_0^2 + \beta^2}}\right) \tag{11.14}$$

which means that the magnification is

$$\mu = \frac{1}{4}\left\{\frac{\beta}{\left(4\alpha_0^2 + \beta^2\right)^{1/2}} + \frac{\left(4\alpha_0^2 + \beta^2\right)^{1/2}}{\beta} \pm 2\right\} \tag{11.15}$$



Note the limits: as $\beta \longrightarrow \infty$, $\mu_+ \longrightarrow 1$ and $\mu_- \longrightarrow 0$; in other words, there is no lensing. On the other hand, when $\beta \longrightarrow 0$, both μ_+ and μ_- go to infinity. As you can see from the curves, when $\beta < \alpha_0$, gravitational lens magnification is important.

There are a few other relations that should be kept in mind. Since the magnification of a lens is given by

$$\mu = \frac{1}{4}\left\{\frac{\beta}{\left(4\alpha_0^2+\beta^2\right)^{1/2}} + \frac{\left(4\alpha_0^2+\beta^2\right)^{1/2}}{\beta} \pm 2\right\} \tag{11.15}$$

the brightness ratio of two images is

$$\nu = \frac{\mu_+}{\mu_-} = \left\{\frac{\left(4\alpha_0^2+\beta^2\right)^{1/2}+\beta}{\left(4\alpha_0^2+\beta^2\right)^{1/2}-\beta}\right\}^2 \tag{11.16}$$

which simplies through (11.12) to

$$\nu = \frac{\mu_+}{\mu_-} = \left(\frac{\theta_1}{\theta_2}\right)^2 \tag{11.17}$$

Some other convenient relations (which can be proved via simple algebra) are

$$\theta_1 - \theta_2 = \left(4\alpha_0^2 + \beta^2\right)^{1/2} \tag{11.18}$$

$$\beta = \theta_1 + \theta_2 \tag{11.19}$$

and

$$\theta_1 \theta_2 = -\alpha_0^2 \tag{11.20}$$

(Again, remember, that the angles are vector quantities, so angles such as θ can be positive or negative.)

General Scale Lengths for Lenses

It's useful to keep in mind some of the numbers that are typical of gravitational lens problems. Remember that the key value is α_0,

$$\alpha_0 = \left(\frac{4G\mathcal{M}}{c^2} \frac{D_{ds}}{D_d D_s} \right)^{1/2} \tag{11.09}$$

For the MACHO project, which tries to detect stars in the Large Magellanic Cloud ($D_s = 50$ kpc) that are lensed by stellar-type objects in the Milky Way,

$$\alpha_0 = 4 \times 10^{-4} \left(\frac{\mathcal{M}}{\mathcal{M}_\odot} \right)^{1/2} \left(\frac{D_{ds}}{D_d} \right)^{1/2} \text{ arcsec} \tag{11.21}$$

In this case, the image separations $\theta \sim \alpha_0$ are sub-milliarcsec, so you will never be able to resolve the two separately lensed images. But, with both images landing on top of each other, the source can be (greatly) magnified.

For quasar lensing by a typical large elliptical galaxy ($10^{12}\mathcal{M}_\odot$) in an Einstein-de Sitter universe,

$$\alpha_0 = 1.6 \left(\frac{\mathcal{M}}{10^{12}\mathcal{M}_\odot} \right)^{1/2} \left(\frac{z_s - z_d}{z_s z_d} \right)^{1/2} h^{1/2} \text{ arcsec} \tag{11.22}$$

where h is the Hubble Constant in units of 100 km/s/Mpc. Here, the typical scale-length is a little over an arcsec, so two separate images can be resolved.

Gravitational Lens Time Delay

If the source behind a gravitational lens is variable, then, of course, the images we see will also be variable. However, the images will not necessarily vary at the same time: in general, there will be a *time delay* between the two events. This comes from two components.

First, there is a purely geometrical time delay. Consider the value of $D_{ds} + D_d - D_s$ for each image of the lens. From $\triangle OSI$ and the law of cosines

$$D_s^2 = D_d^2 + D_{ds}^2 - 2 D_d D_{ds} \cos(180 - \alpha) \tag{11.23}$$

If we substitute

$$\cos(180 - \alpha) = -\cos\alpha = -\left\{1 - 2\sin^2(\alpha/2)\right\} \tag{11.24}$$

then

$$D_s^2 = D_d^2 + D_{ds}^2 + 2 D_d D_{ds} - 4 D_d D_{ds} \sin^2(\alpha/2) \tag{11.25}$$

or

$$\begin{aligned} 4 D_d D_{ds} \sin^2(\alpha/2) &= \left(D_d^2 + D_{ds}^2\right)^2 - D_s^2 \\ &= (D_d + D_{ds} - D_s)(D_d + D_{ds} + D_s) \end{aligned} \tag{11.26}$$

If we now use the law of sines,

$$\frac{\sin\alpha}{D_s} = \frac{\sin(\theta - \beta)}{D_{ds}} \tag{11.27}$$

and consider that all the angles are small, so that $\sin\alpha \approx \alpha$, $\sin(\theta - \beta) \approx \theta - \beta$, $\sin^2\alpha/2 \approx \alpha^2/4$, and $D_d + D_{ds} + D_s \approx 2D_s$, then

$$D_d + D_{ds} - D_s = \frac{4D_d D_{ds} \sin^2(\alpha/2)}{D_d + D_{ds} + D_s}$$
$$= \frac{D_d D_{ds} \alpha^2}{2D_s} \tag{11.27}$$

Finally, by substituting for α using (11.07), we have

$$D_d + D_{ds} - D_s = \frac{D_d D_s}{2D_{ds}} \left(\vec{\theta} - \vec{\beta}\right)^2 = c\,\Delta t \tag{11.28}$$

Since there are two values of $\vec{\theta}$, there are two values for the geometrical time delay. So the delay of one image with respect to the other (which is all we can observe), is

$$\Delta T_g = \frac{D_d D_s}{2cD_{ds}} \left\{ (\theta_1 - \beta)^2 - (\theta_2 - \beta)^2 \right\} \tag{11.29}$$

If you now recall that

$$\alpha_0 = \left(\frac{4G\mathcal{M}}{c^2} \frac{D_{ds}}{D_d D_s} \right)^{1/2} \tag{11.09}$$

and

$$\beta = \theta - \frac{\alpha_0^2}{\theta} \tag{11.10}$$

then, we can substitute these in to get

$$\Delta T_g = \frac{1}{2c} \frac{4G\mathcal{M}}{c^2 \alpha_0^2} \left\{ \frac{\alpha_0^4}{\theta_1^2} - \frac{\alpha_0^4}{\theta_2^2} \right\}$$
$$= \frac{2G\mathcal{M}}{c^3} \alpha_0^2 \left\{ \frac{\theta_2^2 - \theta_1^2}{\theta_1^2 \theta_2^2} \right\} \tag{11.30}$$

But, we also have the relation

$$\theta = \frac{1}{2}\left(\beta \pm \sqrt{4\alpha_0^2 + \beta^2}\right) \tag{11.12}$$

which gives

$$\theta_1\theta_2 = -\alpha_0^2 \tag{11.20}$$

So, with this substitution, we get the final expression for the geometrical time delay from one image to the other.

$$\Delta T_g = \frac{2G\mathcal{M}}{c^3}\left\{\frac{\theta_1^2 - \theta_2^2}{|\theta_1\theta_2|}\right\} \tag{11.31}$$

The second source of signal time delay arises from the passage of the photons through a potential. Any time light passes through a gravitational potential, it experiences the Shapiro time delay

$$\Delta t = -\frac{2}{c^3}\int \Phi dl \tag{11.32}$$

where dl is the photon's path through the potential well. For the case of a weak potential and small deflection, the delay a photon experiences going from source S to point I is

$$\begin{aligned}\Delta t_1 &= -\frac{2}{c^3}\int_0^{D_{ds}} \frac{G\mathcal{M}}{r} dl = -\frac{2G\mathcal{M}}{c^3}\int_0^{D_{ds}} \frac{dx}{(x^2+\xi^2)^{1/2}} \\ &= -\frac{2G\mathcal{M}}{c^3}\ln\left\{x + \sqrt{x^2+\xi^2}\right\}\Bigg]_0^{D_{ds}} = \frac{2G\mathcal{M}}{c^3}\ln\left(\frac{\xi}{2D_{ds}}\right)\end{aligned} \tag{11.33}$$

Similary, the delay in going from point I to the observer is

$$\Delta t_2 = \frac{2G\mathcal{M}}{c^3}\ln\left(\frac{\xi}{2D_d}\right) \tag{11.34}$$

So the total time delay for the potential is

$$\Delta t_p = \Delta t_1 + \Delta t_2 = \frac{2G\mathcal{M}}{c^3}\left\{\ln\left(\frac{\xi}{2D_{ds}}\right) + \ln\left(\frac{\xi}{2D_d}\right)\right\} \tag{11.35}$$

We can now calculate the difference in the time delay from one image to the other, by realizing that at very large distances from the lense, the potential delay is negligible. So, if we pick some distance, $D' \gg \xi$, but $D' \ll D_{ds}, D_s$, then we can write (11.35) as

$$\Delta t_p = \frac{2G\mathcal{M}}{c^3}\left\{\ln\left(\frac{\xi}{2D'}\right) + \ln\left(\frac{D'}{D_{ds}}\right) + \ln\left(\frac{\xi}{2D'}\right) + \ln\left(\frac{D'}{D_d}\right)\right\} \tag{11.36}$$

The second and fourth terms do not depend on ξ and are the same for both images. So, when we take the difference of the two time delays we get

$$\Delta T_p = \frac{4G\mathcal{M}}{c^3}\left\{\ln\left(\frac{\xi_i}{2D'}\right) - \ln\left(\frac{\xi_2}{2D'}\right)\right\} \tag{11.37}$$

If we then substitute for ξ using $\theta = \xi/D_d$ and note that D' is the same for both rays, we get the final expression for the difference between the two Shapiro delays

$$\Delta T_p = \frac{4G\mathcal{M}}{c^3}\ln\left(\left|\frac{\theta_1}{\theta_2}\right|\right) \tag{11.38}$$

The total time delay of one image with respect to the other is then

$$\Delta T = \Delta T_g + \Delta T_p = \frac{4G\mathcal{M}}{c^3}\left\{\frac{\theta_1^2 - \theta_2^2}{2|\theta_1\theta_2|} + \ln\left(\left|\frac{\theta_1}{\theta_2}\right|\right)\right\} \tag{11.39}$$

Note, however, that this equation neglects two important effects.

1) The universe is expanding and possibly non-Euclidean. Thus, the expressions for distance depend on q_0, H_0, and z in some complex manner. (They are angular size distances.)
2) Cosmological lenses are not point sources. The best lenses are elliptical galaxies (which are fairly concentrated collections of stars), but these galaxies are located inside of clusters. So real lenses include the effects of both the galaxy and the cluster (which, of course, add vectorially).