A single-domain spectral method for black hole puncture data

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1. Introduction (1)

- The puncture method provides general relativistic initial data corresponding to single and binary black hole sources.
- Using the Bowen-York extrinsic curvature, only the Hamiltonian constraint needs to be solved.
- By an appropriate splitting of the conformal factor one obtains an equation that determines an auxiliary function u which is well-defined on the entire space $\mathbb{R}^3$.
- In Cartesian coordinates, u is merely $\mathcal{C}^2$-differentiable at the location of the punctures.
- Therefore, when working on Cartesian grids, a spectral method gives only a fourth-order algebraic convergence.

1. Introduction (2)

- In this talk we demonstrate how specific coordinate transformations render the puncture data smooth.
- Consequently, these data can be obtained by means of a single-domain spectral method which exhibits exponential convergence and leads to highly accurate solutions.
- As an application we investigate small mass ratios of binary black holes and compare these with the corresponding test mass limit that can be obtained through a semi-analytical limiting procedure.

2. The Spectral Method (1)

- **The auxiliary function u satisfies an elliptic equation of the following form:**

$$\triangle u + \varrho(u) = 0$$

- **This equation is considered in specific coordinates (A, B, φ) with**

$$A \in [0,1], \quad B \in [-1,1], \quad \varphi \in [0, 2\pi),$$

in which u is *regular* within the entire spatial domain.

- **In particular, spatial infinity corresponds to:** $A = 1$
- **Since u obeys a physical fall-off condition at infinity, we consider an additional function U given by:**

$$u = (A - 1)U$$

2. The Spectral Method (2)

- For the function U, we perform a Chebyshev expansion with respect to the coordinates A and B, and a Fourier expansion with respect to φ.
- We solve the corresponding discretized version of Hamiltonian constraint by means of a Newton-Raphson scheme.
- No boundary conditions need to be imposed. They are replaced by regularity and periodicity conditions which are realized automatically.
- We evaluate the convergence of the spectral method by computing the 'global relative accuracy' defined by

$$\delta_{n,m}(U) = \mathbf{max}_{(A,B,\varphi)}|1 - U_n/U_m|$$

U_n denotes a specific nth order spectral approximation of the function U.

3. Puncture initial data (1)

- **The puncture method provides conformally flat and maximally sliced initial data.**
- **These data possess specific singularities at the locations of the ‘punctures’.**
- **Taking a spacetime consisting of N_p punctures, the singular terms of the conformal factor are split off:**

$$\psi = 1 + \sum_{n=1}^{N_p} \frac{m_n}{2|\boldsymbol{x}_n|} + u$$

$\boldsymbol{x}_n$**: distance vector with respect to the nth puncture;**

m_n**: ‘bare mass’ of the nth puncture;**

$u \to 0$ **as** $|\boldsymbol{x}_n| \to \infty$

- **In general, u is $\mathcal{C}^2$-differentiable at the punctures (in Cartesian coordinates).**

3.1 Single-puncture initial data (1)

- **For a single puncture, we apply the following coordinate transformation:**

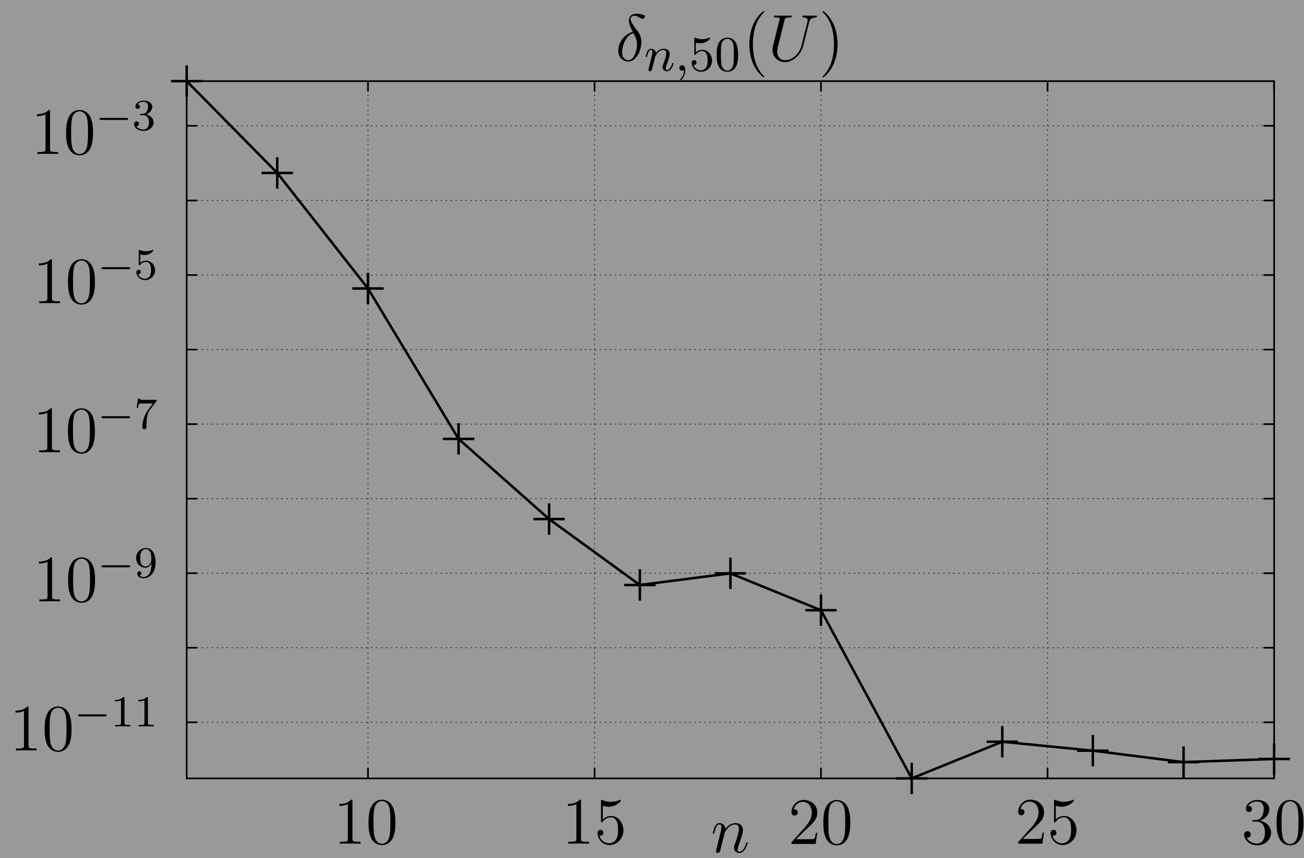
$$A = \left(1 + \frac{m}{2r}\right)^{-1}, \quad B = 2\theta/\pi - 1, \quad \varphi = \phi$$

(r, θ, ϕ)**: spherical coordinates about the puncture**

- **Our spectral method yields solutions to the Hamiltonian constraint that are regular at the puncture.**

3.1 Single-puncture initial data (2)

- **Representative example demonstrating an exponential convergence of the spectral method:**



Puncture with vanishing linear momentum;
Spin $S^i = m^2 w\, \delta^i_1$ with $w = 0.2$

3.2 Two-puncture initial data (1)

- **The two punctures be located on the x-axis at $x_\pm = \pm b$.**
- **The single-puncture data suggest the introduction of coordinates in which the distances $r_\pm$ are regular where:**

$$r_\pm = \sqrt{(x \mp b)^2 + \rho^2}$$

(x, ρ, φ)**: cylindrical coordinates about the x-axis**

- **Example: In two dimensions, the transformation**

$$c = C^2, \quad c = x + \mathbf{i}y, \quad C = X + \mathbf{i}Y$$

renders the distance smooth with respect to X and Y:

$$\sqrt{x^2 + y^2} = \sqrt{c\bar{c}} = C\bar{C} = X^2 + Y^2$$

- **This transformation maps a right angle at the origin to a straight line through the origin.**

3.2 Two-puncture initial data (2)

- For the two-puncture situation, we obtain a similar effect on angles with the transformation (see figure next slide):

$$c = \frac{b}{2}\left(C + C^{-1}\right), \quad c = x + \mathbf{i}\rho, \quad C = X + \mathbf{i}R$$

Location of the punctures: $C_\pm = \pm 1$

- Moreover, we get that $r_\pm$ is regular in X and R at $C_\pm$:

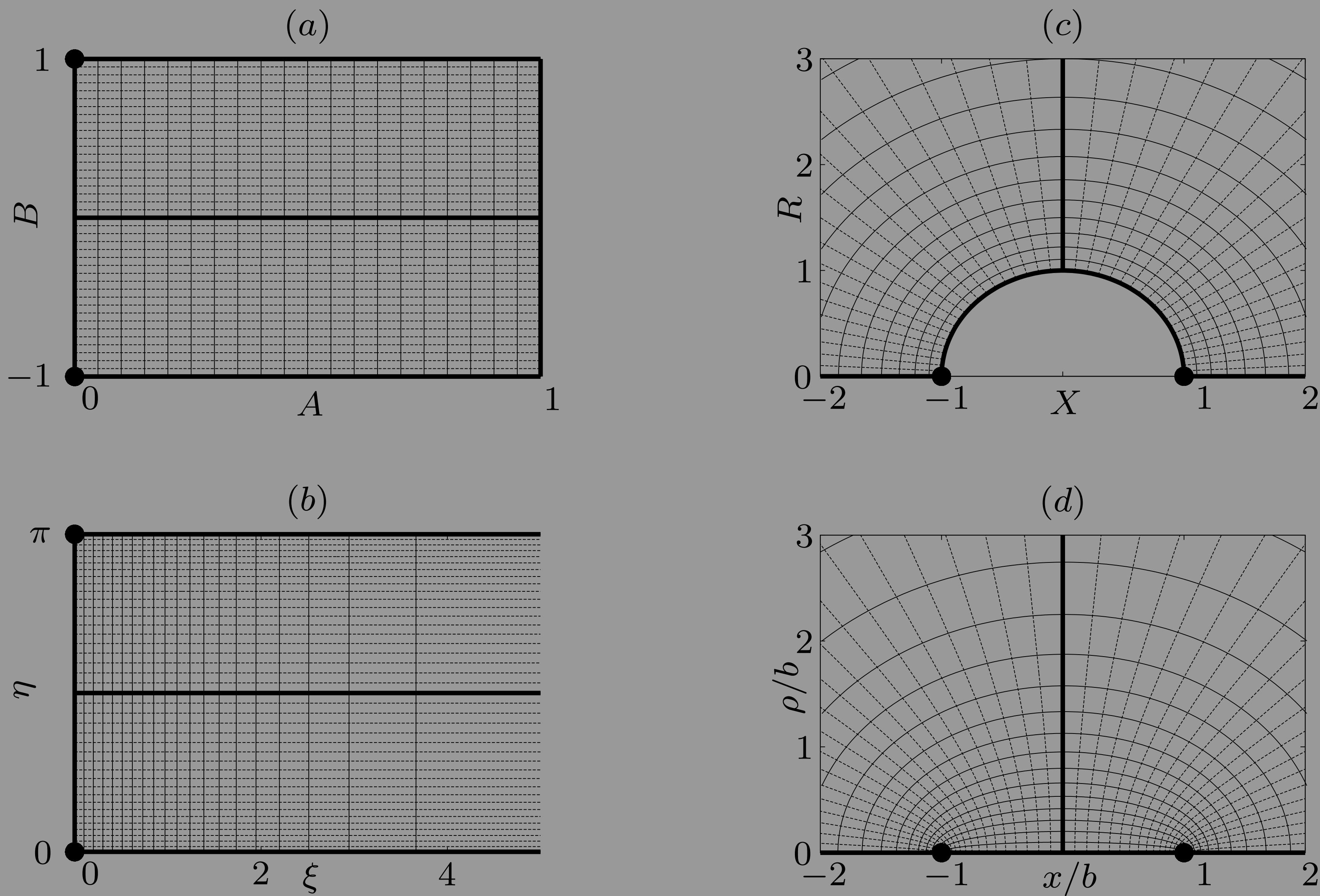
$$r_\pm = |c \mp b| = \frac{b}{2\sqrt{X^2 + R^2}}\left[(X \mp 1)^2 + R^2\right]$$

- The mapping of the space of (X, R)-coordinates to our single domain of (A, B)-coordinates is done as follows (φ remains unchanged):

$$1. \quad C = \mathbf{e}^{\zeta}, \quad \zeta = \xi + \mathbf{i}\eta \quad \text{prolate spheroidal coordinates}$$

$$2. \quad \xi = 2\,\mathbf{artanh}\, A\,, \quad \eta = \frac{\pi}{2} + 2\mathbf{arctan}\, B$$

3.2 Two-puncture initial data (3)

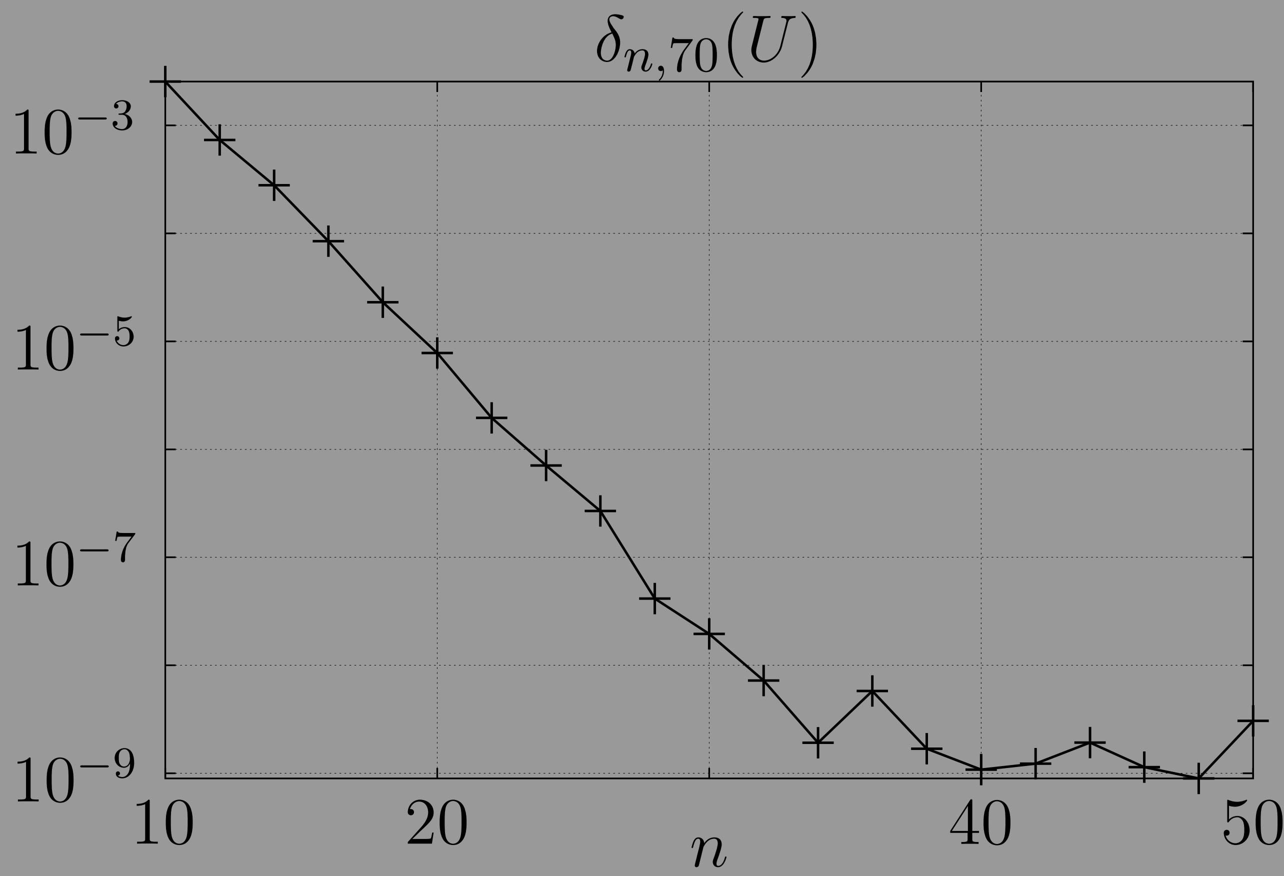


Several coordinate patches for the two puncture initial data problem.

The punctures are indicated by bullets.

3.2 Two-puncture initial data (4)

- **Representative example demonstrating an exponential convergence of the spectral method:**



Two punctures with vanishing spins and equal bare masses, $m_+ = m_- = b$**;**
Linear momenta: $P^i_\pm = \pm 0.2\, b\, \delta^i_2$

3.3 Test mass limit (1)

- **An analytic study shows that performing the test mass limit $m_-/m_+ \to 0$ can be reduced to the solution of a modified single-puncture initial data problem.**
- **We found that for ratios $m_-/m_+ \geq 10^{-3}$ the spectral scheme yields reliable results that approach those of the test mass limit.**
- **In particular, for mass ratios of 10^{-3}, we obtained four digits of accuracy with our exponentially converging two-puncture calculations.**

3.3 Test mass limit (2)

- **We discuss an example, for which the distance $D = 2b$ and the test particle's velocity v_- obey the relations valid for the last stable circular orbit in the gravitational field of a Schwarzschild black hole of mass m_+:**

$$\frac{D}{m_+} = \frac{5}{2} + \sqrt{6}\,, \quad v_- = \frac{4\sqrt{3}}{5 + 2\sqrt{6}}$$

m_-/m_+	u_-	$2Du_+/m_-$	$\lim_{r\to\infty}(2ru/m_-)$
10^{-1}	**0.03417**	**0.2011**	**0.1688**
10^{-2}	**0.03406**	**0.1635**	**0.1601**
10^{-3}	**0.03406**	**0.1596**	**0.1592**
0	**0.0340568**	**0.159094**	**0.159094**

4. Conclusions (1)

- **Puncture data can be rendered smooth by introducing appropriate coordinates.**
- **In particular, our transformation maps the entire $\mathbb{R}^3$ onto a single rectangular domain with the punctures at the boundary.**
- **We have demonstrated the expected exponential convergence of our single-domain spectral method.**
- **As an application small mass ratios have been considered, and the corresponding results approach the test mass limit.**
- **The study of specific coordinate transformations might also help in reducing the number of domains that are used by methods for binary black hole excision data.**
- **We have started a corresponding investigation based on a coordinate transformation that requires two coordinate patches.**