Absorbing Boundary Conditions

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Overview

Appproach: (Engquist/Majda 1977)

- **Wave-type equations**

$$Lu = \left(\left(\frac{\partial^2}{\partial x^2} + g\,(x,y)\,\frac{\partial^2}{\partial y^2} + b\,(x,y)\,\frac{\partial}{\partial x} \right) - \frac{\partial^2}{\partial t^2} \right) u = 0$$

  - ⋆ factorize *L* (theory of pseudo-differential operators):

$$L \cong \left(\frac{\partial}{\partial x} + \lambda_+ \left(x, y, \frac{\partial}{\partial y}, \frac{\partial}{\partial t} \right) \right) \left(\frac{\partial}{\partial x} - \lambda_- \left(x, y, \frac{\partial}{\partial y}, \frac{\partial}{\partial t} \right) \right)$$

  - ⋆ write the perfect absorbing bc as:

$$\left(\frac{\partial}{\partial x} - \lambda_- \left(x, y, \frac{\partial}{\partial y}, \frac{\partial}{\partial t} \right) \right) \bigg|_{x=0} = 0$$

  - ⋆ consider the symbol of the operator λ_- and make approximations to it
  - ⋆ come back (from the symbol notation to the operators), get by replacing:

$$\omega \text{ with} \frac{1}{i}\frac{\partial}{\partial t} \text{ and } k_y \text{ with} \frac{1}{i}\frac{\partial}{\partial y}$$

the approximations of the operator λ_- and use them as boundary conditions

- **General I order evolution systems**

$$\frac{\partial u}{\partial t} = \left(A_1 \frac{\partial}{\partial x} + A_2 \frac{\partial}{\partial y} + B \right) u$$

⋆ rewrite the system as:

$$\frac{\partial u}{\partial x} = \left(A \frac{\partial}{\partial t} + E \frac{\partial}{\partial y} + \tilde{B} \right) u$$

where $A = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix}$, $\Lambda_1 < 0$ and $\Lambda_2 > 0$

⋆ make a transformation

$$w = V \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial y}, x, y \right) u$$

so that

$$\frac{\partial w}{\partial x} = \begin{pmatrix} \Lambda_{11} \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial y}, x, y \right) & 0 \\ \Lambda_{21} \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial y}, x, y \right) & \Lambda_{22} \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial y}, x, y \right) \end{pmatrix} w$$

where $\Lambda_{11} (0, 1, x, y) = \Lambda_1$ and $\Lambda_{22} (0, 1, x, y) = \Lambda_2$

⋆ write the perfect absorbing bc as:

$$P_{ing} w = 0$$

where P_{ing} gives the components of w coresponding to the negative e-values of Λ_{11}

⋆ get the 'local' approximations for the operator P_{ing} as in the previous case

Absorbing Boundary Conditions for scalar wave equation

Consider the wave equation in 3-dimensional Minkowski space for the field u:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \ .$$

Assume u to correspond to *data having support in the half-plane* $x \geq 0$.

- **obtaining the boundary conditions:**
  1. Factorize the laplacian:

$$L \cong \left(\frac{\partial}{\partial x} + \sqrt{\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}} \right) \left(\frac{\partial}{\partial x} - \sqrt{\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}} \right)$$

  2. Write perfect absorbing bondary condition :

$$\lambda_- \left(\frac{\partial}{\partial y}, \frac{\partial}{\partial t} \right) = \left(\frac{\partial}{\partial x} - \sqrt{\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}} \right) \Bigg| u_{x=0} = 0$$

...The boundary condition is *non local* in space and in time!

3. perform Taylor expansion of the symbol:

$$\lambda_{-}\left(ik_y, i\omega\right) = i\omega\sqrt{1-\frac{k_y^2}{\omega^2}} \approx i\omega\left(1-\frac{k_y^2}{2\,\omega^2}-\frac{k_y^4}{8\,\omega^4}+\dots\right)$$

... perfect absorbtion at normal incidence ($k_y = 0$)!

4. Back to the operators and get a sequence of 'local' boundary conditions:

$$\left.\left(\frac{\partial u}{\partial x}-\frac{\partial u}{\partial t}\right)\right|_{x=0} = 0 \text{ , \textbf{(I approx)}}$$

$$\left.\left(\frac{\partial^2 u}{\partial t\,\partial x}-\frac{\partial^2 u}{\partial t^2}+\frac{1}{2}\frac{\partial^2 u}{\partial y^2}\right)\right|_{x=0} = 0 \text{ , \textbf{(II approx)}}$$

$$\left.\left(\frac{\partial^4 u}{\partial t^3\,\partial x}-\frac{\partial^4 u}{\partial t^4}+\frac{1}{2}\frac{\partial^4 u}{\partial t^2\,\partial y^2}+\frac{1}{8}\frac{\partial^4 u}{\partial y^4}\right)\right|_{x=0} = 0 \text{ , \textbf{(III approx)}}\,etc.$$

5. Making use of the wave equation the previous relations, we can be put them in the form:

$$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right) u \Bigg|_{x=0} = 0 \text{ , **(I approx)**}$$

$$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right)^2 u \Bigg|_{x=0} = 0 \text{ , **(II approx)**}$$

$$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right)^2 \left[\frac{\partial^2}{\partial t^2} - \frac{1}{4}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)^2\right] u \Bigg|_{x=0} = 0 \text{ , **(III approx)**} etc.$$

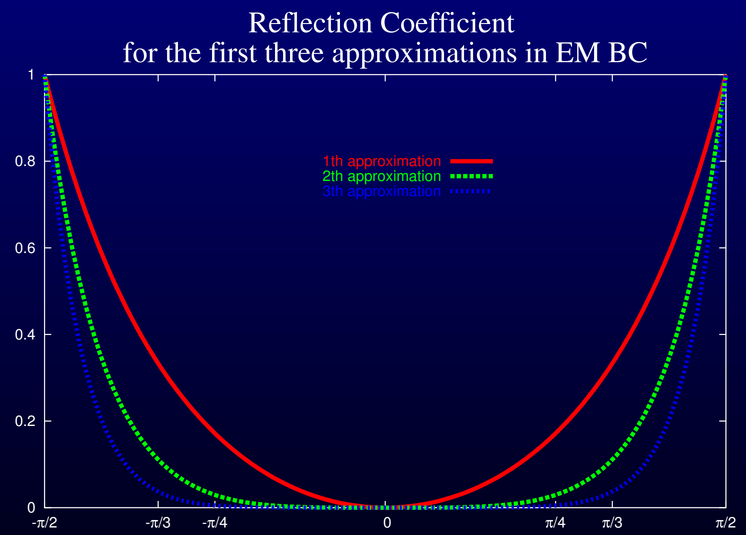
- **Results–Engquist/Majda 1977**
  - ⋆ well-posedness:
    1. By an *energy estimate* they show the first order boundary condition leads to a well-posed problem
    2. Using a theorem of Kreiss they conclude that the second order boundary condition also leads to a well-posed problem
    3. The third approximation (4'th order) is ill-posed
  - ⋆ Reflection of the outgoing modes
    1. normal incidence: no reflection
    2. at 45°: reflection coefficients are: 0.17, 0.029, 0.0055
    3. close to glancing: all the reflection coefficients tend to unity.

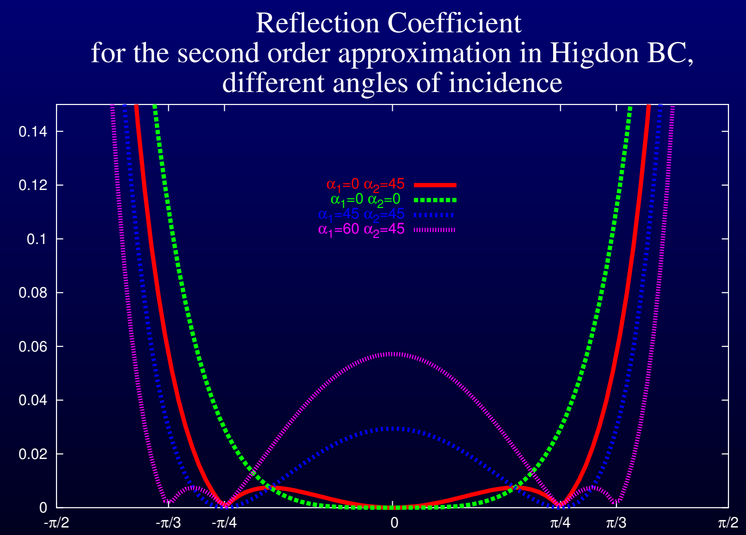
- **Generalization** A more *general form* of the EM boundary conditions was found by Higdon(1986) (appropriate for dispersive problems)

$$\left[\prod_{j=1}^{p}\left((\cos\alpha_j)\frac{\partial}{\partial t}-c_j\frac{\partial}{\partial x}\right)\right]u=0$$

- **Reflection coefficient** (for a plane wave traveling with wave speed *c* and hitting the x-boundary at angle of incidence $\cos\theta$)

$$R=\prod_{j=1}^{p}\left|\frac{c\cos\alpha_j-c_j\cos\theta}{c\cos\alpha_j+c_j\cos\theta}\right|$$





Sommerfeld Boundary Condition

Assuming that initial data is *spherically symmetric* with compact support in the computational domain, at the edge of the grid the wave will be purely outgoing, of the form $u = \frac{f(r-t)}{r}$. Then the *exact* boundary condition for this system will be the radiative BC (Sommerfeld BC):

$$(\partial_r + \partial_t)(ru)|_{boundary} = 0$$

- Radiation BC is just an absorbing bc in I order approximation applied in the *radial direction*
- This boundary condition is appropriate for radiation problems, for which the waveforms will approach spherical symmetry, about the centre of the system at large distances.
- We can introduce the II order radiative bc as:

$$(\partial_r + \partial_t)^2 (ru)\Big|_{boundary} = 0$$

- Now projecting them on the x-direction it will give:

$$\frac{x}{r}\partial_t u + \partial_x u + \frac{x}{r^2}u = 0$$

$$\frac{x}{r}\partial^2_{tx}u + \partial^2_x u + \frac{x^2}{r^3}\partial_t u + \frac{x}{r}\left(\frac{3}{r} - \frac{r}{x^2}\right)\partial_x u = 0$$

Numerical Tests of Absorbing Boundary Conditions for Scalar Wave Equation

Evolution equations and initial data:

$$(\partial_{tt} - \triangle)\, u = 0$$

$$\begin{aligned} u(x,y,z,0) &= f(x,y,z) \\ \partial_t u(x,y,z,0) &= g(x,y,z) \end{aligned}$$

implementation: leap-frog scheme:

$$\begin{aligned} \phi^{n+1}_{ijk} = \quad & 2\phi^n_{ijk}\left(1 - \rho_x^2 - \rho_y^2 - \rho_z^2\right) - \rho_x^2\left(\phi^n_{i+1jk} - \phi^n_{i-1jk}\right) \\ & \rho_y^2\left(\phi^n_{ij+1k} - \phi^n_{ij-1k}\right) - \rho_z^2\left(\phi^n_{ijk+1} - \phi^n_{ijk-1}\right) - \phi^{n-1}_{ijk} \end{aligned}$$

(ρ_x,ρ_y,ρ_z-Courant factors)

Absorbing Boundary Conditions

$$(\partial_x - \cos\alpha\partial_t)|_{x-boundary}\, u = 0 \qquad \text{(Abs I)}$$

$$(\partial_x - \cos\alpha_1\partial_t)\,(\partial_x - \cos\alpha_2\partial_t)|_{x-boundary}\, u = 0 \qquad \text{(Abs II)}$$

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- The relations (Abs I) and (Abs II) are the 1st and 2nd order absorbing BC for a *plane wave* hiting the boundary with the angle α in the case (Abs I) and with the angle α_1 or α_2 for (Abs II).

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- We compare these with the radiative boundary conditions (absorbing bc in the radial direction):

$$\left(\partial_r + \partial_t\right)(ru)\Big|_{boundary} = 0 \qquad \text{(Rad I)}$$

$$\left(\partial_r + \partial_t\right)^2 (ru)\Big|_{boundary} = 0 \qquad \text{(Rad II)}$$

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- The relations (Rad I) and (Rad II) are the perfect absorbing BC for a *spherical wave* propagating radially.

Global quantities used to classify the BC

- **Energy**

$$E(t) = \int\int\int \left(\left(\frac{\partial u}{\partial t} \right)^2 + |\bigtriangledown u|^2 \right) dV$$

- **Norms of the reflected wave**
  - ⋆ Infinity norm:

$$||\phi||_\infty = \max \left| \phi_{ijk} - \phi_{ijk}^{exact} \right|$$

  - ⋆ L2-norm:

$$||\phi||_2 = \sqrt{\sum \left| \phi_{ijk} - \phi_{ijk}^{exact} \right|^2 /N}$$

  - ⋆ Soboloev Norm:

$$||\phi||_S = \sum \left(\left| \bigtriangledown \left(\phi_{ijk} - \phi_{ijk}^{exact} \right) \right|^2 \right) /N$$

where ϕ_{ijk} is a grid variable and ϕ_{ijk}^{exact} is its exact value.

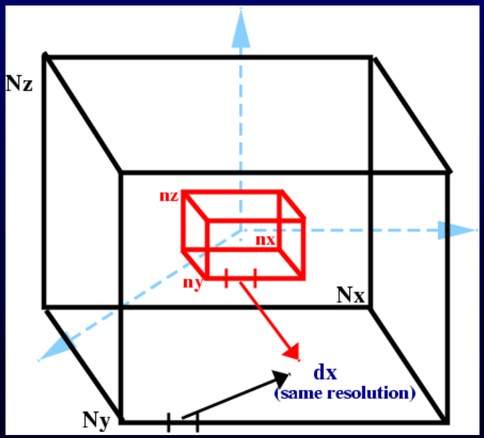
Calculation of ϕ_{ijk}^{exact}

In order to obtain a measure of the reflected wave, we perform tests on two grids: A small grid, and a much larger grid which contains the small grid.

We perform the runs for as long as the small grid is causally disconnected from the large grid boundaries.

For this period of time, the solution on the large grid is regarded as the exact solution, uncontaminated by boundary effects: ϕ_{ijk}^{exact}

We can compare the solution on the small grid to ϕ_{ijk}^{exact} to determine the quality of boundary conditions.



Results

We are interested to see how these boundary conditions behave for different types of initial data:

1. **spherically symmetric pulse, centered at the origin** ID
   - gaussian
2. **spherically symmetric pulse, and off-centered** ID
   - gaussian off-centered
3. **nonspherically symmetric pulse** ID
   - gaussian*ReY22

Grid parameters:

$$\begin{cases} nx = ny = nz = 41 \qquad small \;\; grid \\ \\ Nx = Ny = Nz = 801 \qquad big \;\; grid \end{cases}$$

$dx = dy = dz = 0.001$ $\rho_x = 0.5$; $dt = 0.0005$ ($\sigma = 0.0025$)

$-->$run until t = 10 crossing times = 10×80 iterations = 0.4
(the perturbation from the boundary of the big grid appears at $t = 0.4$)

RESULTS

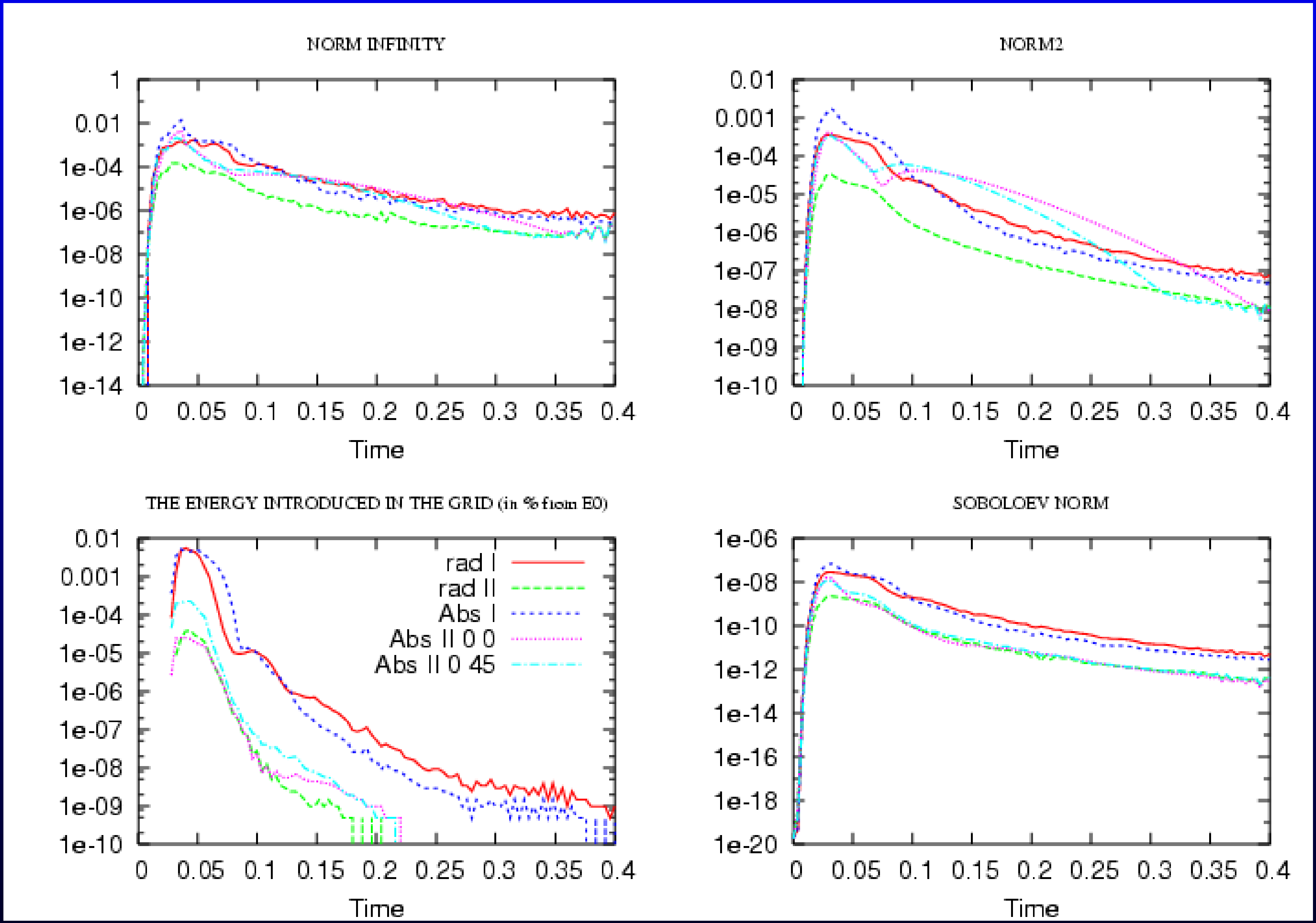


Figure 1: gaussian, centered at origin

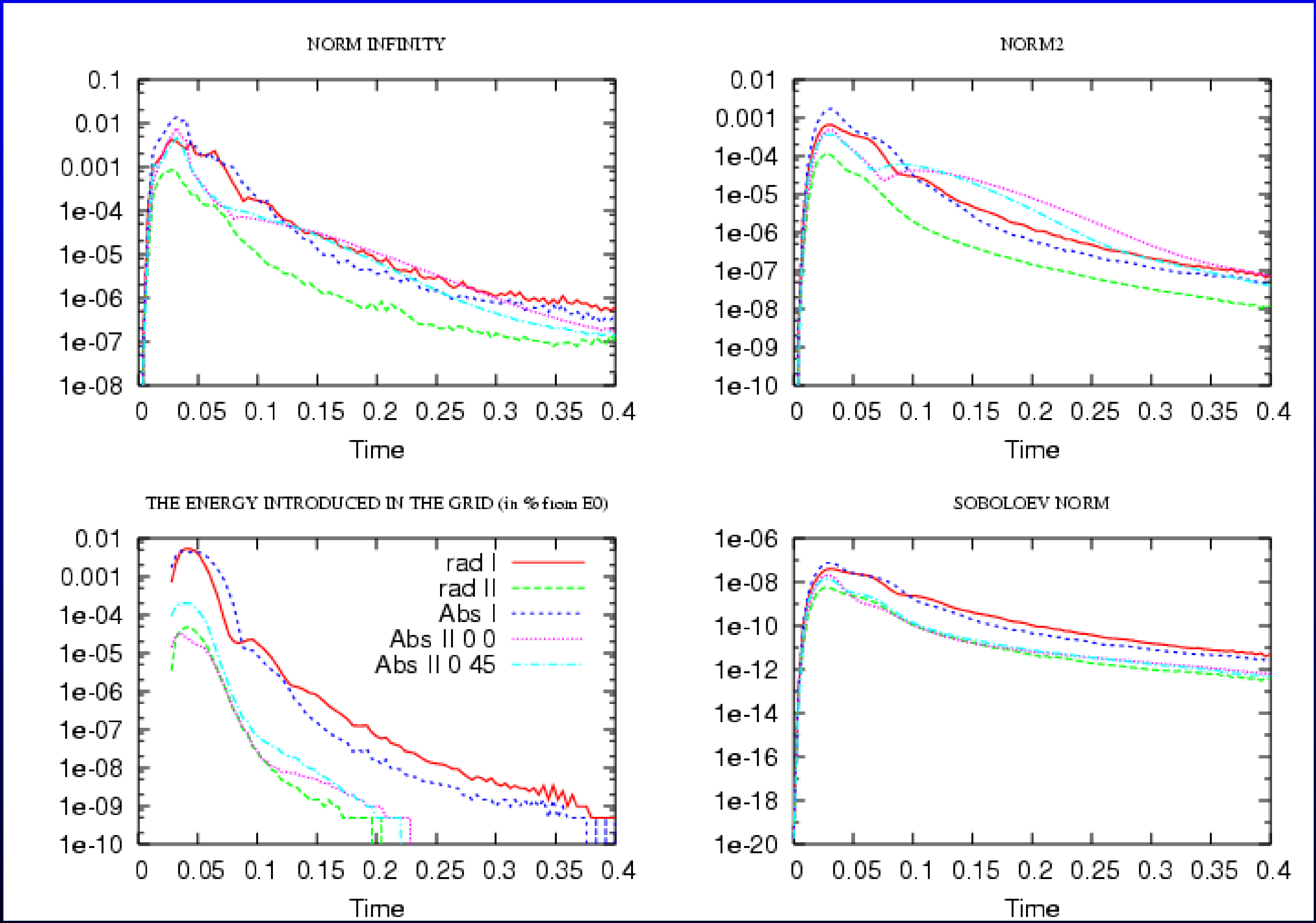


Figure 2: gaussian-offcentered

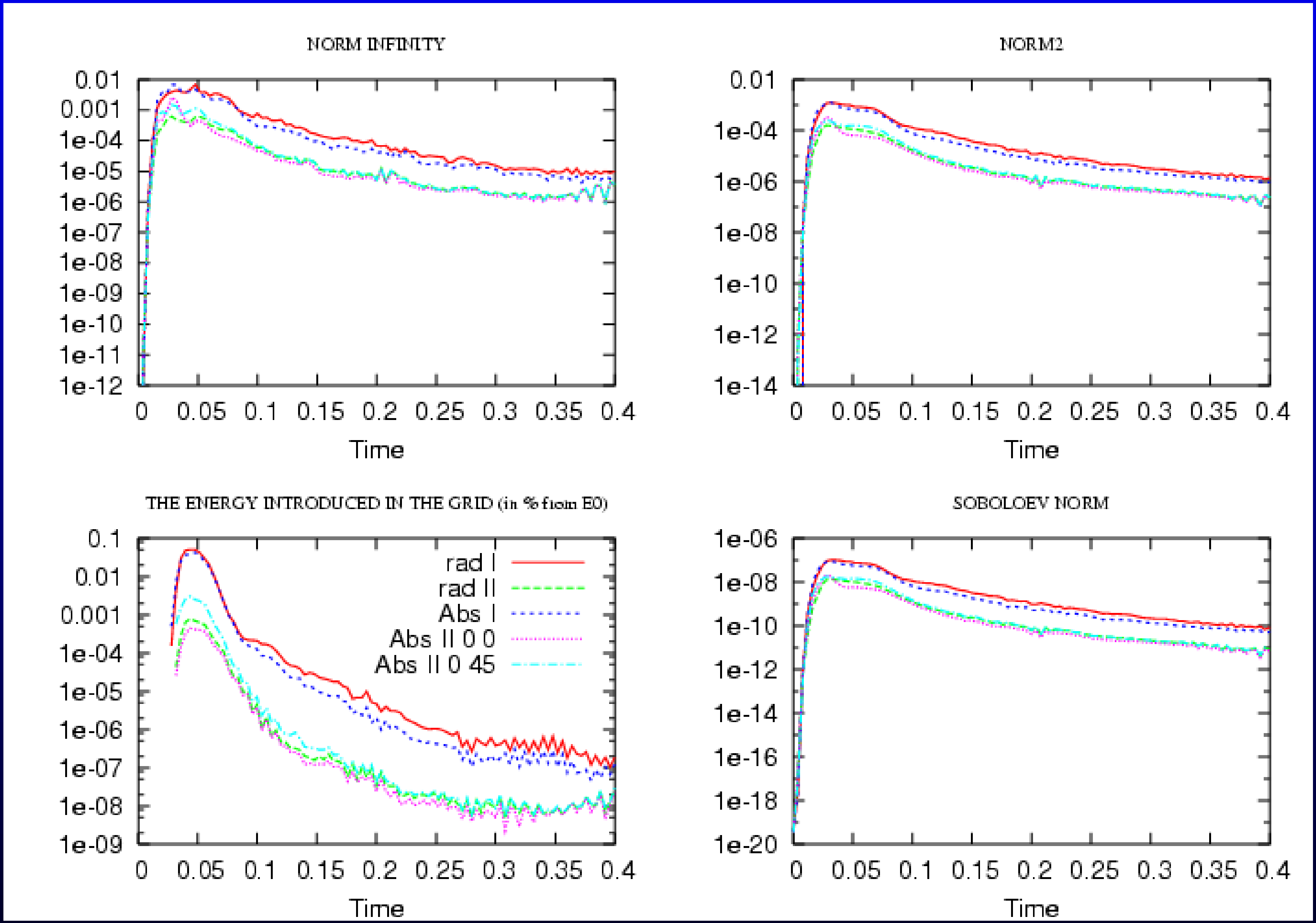


Figure 3: gaussian*ReY22

Conclusions and Outlook

- the second order approximations gave the less reflections in all the cases, especially the second order approximation applied in the radial direction.
- it's worth going to more complicated systems and derive in the same manner absorbing boundary conditions

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3. Find a local differential expression that does not contain any mode parameters and *annihilates* outgoing modes of a preferred (e.g., normal) angle of incidence. Adopt as outgoing boundary condition the vanishing of that condition at the boundary.

appendix

- **symbolic notation of pseudo-differential operators:**

$$b\left(y,t,\frac{\partial}{\partial y},\frac{\partial}{\partial t}\right)u=\int\int e^{i\left(k_y y+\omega t\right)}b\left(y,t,ik_y,i\omega\right)\hat{u}\left(k_y,\omega\right)d\omega dk_y$$

Back