How to make second-order nonlinear BSSN well-posed with boundaries

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- Motivation
- Well-posedness for first-order systems
  Gustafsson, Kreiss & Oliger
- Constraint-preserving boundary conditions
  Calabrese et al (LSU group)
- Second order in space, first order in time
  gr-qc/0402079
- Making the ADM equations well-posed
  gr-qc to come

Motivation I: Types of errors and “instabilities”

a) Getting the wrong solution (eg phase errors)

b) Correct solution in a bad gauge (eg kinks in the slice)

c) Constraint violations (∞ larger solution space!)

or:

1) Perturbation modes with bounded growth

$$||\delta u(\cdot,t)|| \leq f(t)||\delta u(\cdot,0)||$$

— these converge away

2) Perturbation modes that grow arbitrarily rapidly
— show up as grid modes
— destroy convergence

Motivation II: Well-posedness of the continuum problem

- Initial-value problem
  — can use Fourier transform
  — strong hyperbolicity

- Initial-boundary value problem
  — need energy methods
  — symmetric hyperbolicity, for both the main and the constraint system

- Generalize energy methods from first to second order

FIRST ORDER IN SPACE AND TIME

Well-posedness

Start from a system of quasilinear, first-order evolution equations:

$$\partial_t u = P^i(u)\, \partial_i u + S(u)$$

where $u(x^i, t)$ is a vector, P^i square matrices.

Example: The ADM equations reduced to first order by defining $d_{ijk} \equiv \partial_i \gamma_{jk}$.

The problem is well-posed if the solution exists, is unique, and **depends continuously** on the initial data.

Linearization

Linearize around a background solution u_0:

$$\partial_t \delta u = P^i(u_0)\, \partial_i \delta u + Q(u_0)\delta u$$

Well-posedness requires that the solution of the linearized problem is bounded as

$$||\delta u(\cdot, t)|| \leq f(t) ||\delta u(\cdot, 0)||$$

where $f(t)$ does **not** depend on $\delta u(x, 0)$.

Frozen coefficients approximation

We are interested in potential high-frequency instabilities: approximate $P^i(u_0)$ and $Q(u_0)$ as constant.

Q contributes a factor $e^{|Q|t}$ to the bound $f(t)$: neglect it.

This leaves

$$\partial_t \delta u = P^i \partial_i \delta u$$

Fourier transform

Writing u instead of δu now,

$$\partial_t u = P^i \, \partial_i u$$

Fourier transform

$$u(x,t) = \hat{u}(t) e^{i\omega_i x^i}$$

gives an ODE system

$$\partial_t \hat{u} = i|\omega| P_n u$$

where

$$P_n \equiv n_i P^i, \qquad n_i \equiv \frac{\omega_i}{|\omega|}$$

Strong hyperbolicity

If P_n can be diagonalized as $P_n = T \Lambda T^{-1}$ then with $\hat{U} \equiv T^{-1} \hat{u}$

$$\partial_t \hat{U} = i|\omega| \hat{U} \Lambda$$

With Λ real

$$\begin{aligned} |\hat{u}(\omega, t)| &\leq |T| \, |\hat{U}(\omega, t)| \\ &= |T| \, |\hat{U}(\omega, 0)| \\ &\leq |T| \, |T^{-1}| \, |\hat{u}(\omega, 0)| \end{aligned}$$

With T continuous in n_i, and Parseval

$$||u(\cdot, t)|| \leq |T| \, |T^{-1}| \, ||u(\cdot, 0)||$$

Ill-posed systems

With a complex eigenvalue $a \pm ib$,

$$|\widehat{u}(t)| \sim |\widehat{u}(0)| e^{b|\omega|t}$$

With a Jacobi block of size k,

$$|\widehat{u}(t)| \sim |\widehat{u}(0)| (|\omega| t)^{k-1}$$

In either case the blow-up depends on $|\omega|$ and hence on $u(x,0)$.

In numerics, $|\omega|$ is set by the spatial resolution.

Characteristic variables

Fix a unit vector n^i. Decompose $\partial_i = (\partial_n, \partial_A)$.

U is a characteristic variable with respect to n_i with speed λ if

$$\partial_t U_\lambda = \lambda \partial_n U_\lambda + \partial_A \ldots$$

For a first order system

$$U = T^{-1} u$$

with T diagonalizing P_n as before.

Symmetric hyperbolicity

If H exists that is Hermitian, positive definite, **independent** of n_i, and $HP^i = (HP^i)^\dagger$, we have an energy

$$E = \int_\Omega u^\dagger H u \, dV$$

that is conserved

$$\begin{aligned} \frac{dE}{dt} &= \int_\Omega \partial_i (u^\dagger H P^i u) \, dV \\ &= \int_{\partial\Omega} (u^\dagger H P_n u) \, dS \end{aligned}$$

Bound on positive definite energy
$\Rightarrow$ bound on $||u(\cdot, t)||$
$\Rightarrow$ Well-posedness

Maximally dissipative boundary conditions

Simple example: speeds $\pm\lambda$

$$\begin{aligned} E &= \int_\Omega (U_+^2 + U_-^2)\, dV \\ \frac{dE}{dt} &= \int_{\partial\Omega} \lambda (U_+^2 - U_-^2)\, dS \end{aligned}$$

Control E by

$$U_+ = \kappa U_- + \text{free data}, \qquad |\kappa| \le 1$$

Constraint-preserving boundary conditions

Main system and constraint system both symmetric hyperbolic. For each $C_\pm$ there is a $U_\pm$ with

$$C_\pm = \partial_n U_\pm + \partial_A \ldots$$

Formally impose a maximally dissipative boundary condition

$$C_+ = \kappa C_-$$

on the constraint system. $X \equiv U_+ - \kappa U_-$ gives

$$\partial_n X + \partial_A \ldots = 0$$

Boundary system for X

$$\partial_t X = \partial_A \ldots$$

Actually impose

$$U_+ = \kappa U_- + X$$

on the main system. Calabrese et al (LSU group)

FIRST ORDER IN TIME, SECOND ORDER IN SPACE

Strong hyperbolicity

Define complete set of characteristic variables

$$\partial_t U_\lambda = \lambda \partial_n U_\lambda + \partial_A \ldots$$

where now U is made from $\tilde{u} \equiv (u, \partial_i u)$.

This definition of strong hyperbolicity is equivalent to that of Kreiss & Ortiz using a pseudo-differential reduction to first order.

1. Any transversal derivative is a zero speed characteristic variable:

$$\partial_t (\partial_A u) = \partial_A \ldots$$

2. Characteristic variables are unique only up to addition of transversal derivatives:

$$U'_\lambda = U_\lambda + \partial_A \ldots$$

Symmetric hyperbolicity

1. Find a positive definite energy

$$E = \int_\Omega \epsilon(\tilde{u})\, dV$$

conserved in the sense that

$$\partial_t \epsilon = \partial_i F^i$$

2. Find characteristic variables U (not unique).

3. Add $\partial_A u$ terms to U to obtain

$$\frac{dE}{dt} = \int_{\partial\Omega} \lambda(U_+^2 - U_-^2)\, dS$$

(U now unique).

Maximally dissipative and constraint-preserving boundary conditions as before.

ADM form of the Einstein equations

3+1 split of the spacetime metric

$$ds^2 = -\alpha^2\,dt^2 + \gamma_{ij}(dx^i + \beta^i\,dt)(dx^j + \beta^j\,dt)$$

Shorthand notation

$$\partial_0 \equiv \alpha^{-1}(\partial_t - \mathcal{L}_\beta)$$

Evolution equations

$$\begin{aligned}\partial_0\gamma_{ij} &= -2K_{ij}\\ \partial_0 K_{ij} &= -\alpha^{-1}D_iD_j\alpha + R_{ij} - 2K_{il}K^{lj} + KK_{ij}\end{aligned}$$

Constraints

$$\begin{aligned}H &\equiv R - K_{ij}K^{ij} + K^2 = 0\\ M_i &\equiv D_jK^j{}_i - D_iK = 0\end{aligned}$$

NOR modifications of ADM

Principal part of evolution equations is

$$\begin{aligned}\partial_0\gamma_{ij} &\simeq -2K_{ij}\\ \partial_0 K_{ij} &\simeq \frac{1}{2}\gamma^{kl}\left(-\gamma_{ij,kl}-\gamma_{kl,ij}+\gamma_{ki,jl}+\gamma_{kj,il}\right)\end{aligned}$$

1. Densitize the lapse:

$$\alpha = (\det\gamma)^{\sigma/2}\,Q$$

with $\sigma > 0$ and $Q(x^i,t)$ given instead of α.
2. Define auxiliary variables

$$f_i \equiv \gamma^{jk}\gamma_{ij,k}$$

$$\partial_0 K_{ij} \simeq \frac{1}{2}\gamma^{kl}\left(-\gamma_{ij,kl}-(1+\sigma)\gamma_{kl,ij}+f_{i,j}+f_{j,i}\right)$$

3. Use the momentum constraint to evolve f_i:

$$\partial_t f_i \simeq -2\gamma^{jk}K_{ij,k} \simeq -2K_{,i}$$

Strongly hyperbolic in a pseudo-differential sense, and the constraint system too.

(Nagy, Ortiz and Reula 2003)

Our main results

- NOR and its constraint system are also symmetric hyperbolic (in our definition).

- BSSN is symmetric hyperbolic with a densitized lapse and $tr\tilde{A}_{ij} = 0$ and $\det \tilde{\gamma}_{ij} = 1$ imposed continuously.
  — Strongly hyp. enforcing only $tr\tilde{A}_{ij} = 0$.
  — Strongly hyp. modifying equations.

- NOR and BSSN allow constraint-preserving boundary conditions
  — Proof of well-posedness for Dirichlet and Neumann
  — No ill-posed modes for $|\kappa| \leq 1$ (eg Sommerfeld)

Future work

- Numerical testbed Maxwell equations
- Mexico tests and more in full GR
- Why bother with BSSN?
- Modify energy methods to prove well-posedness of the initial-boundary value problem.
- Add hyperbolic or parabolic coordinate conditions (α and β^i are evolved).