

Constraint Preserving Boundary Conditions for linearized BSSN formulation

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Numerical Relativity Lunch
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Outline

- Introduction to BSSN formulation;
- Constraints of the BSSN system;
- Near-flat linearization;
- Boundary conditions for the linearized BSSN system;
- Evolution of the constraints;
- Constraint preserving boundary conditions;

ADM system

Arnowitt–Deser–Misner 3 + 1 decomposition in vacuum: (lapse a ; shift b_i ; 3-metric h_{ij} ; extrinsic curvature k_{ij} ; spatial Ricci tensor R_{ij})

$$\partial_t h_{ij} = -2ak_{ij} + 2D_{(i}b_{j)},$$

$$\partial_t k_{ij} = a[R_{ij} + k_l^l k_{ij} - 2k_{il}k_j^l] + b^l D_l k_{ij} + k_{il} D_j b^l + k_{lj} D_i b^l - D_i D_j a,$$

$$R_i^i + (k_i^i)^2 - k_{ij}k^{ij} = 0, \quad \text{Hamiltonian const.}$$

$$D^j k_{ij} - D_i k_j^j = 0 \quad \text{momentum const.}$$

$$R_{ij} = \frac{1}{2}h^{pq}(\partial_p \partial_j h_{iq} + \partial_i \partial_q h_{pj} - \partial_p \partial_q h_{ij} - \partial_i \partial_j h_{pq}) \\ + h^{pq}h^{rs}(\Gamma_{ipr}\Gamma_{qjs} - \Gamma_{pqr}\Gamma_{ijs}),$$

$$\Gamma_{ijl} = \frac{1}{2}(\partial_i g_{lj} + \partial_j g_{il} - \partial_l g_{ij}).$$

Ricci Tensor is difficult!

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BSSN: the Trace of k_{ij}

The evolution equation for $k = k_l^l$ is simple

$$(1) \quad (\partial_t - b^l D_l)k = a k^{pq} k_{pq} - D^l D_l a$$

(after Hamiltonian constraint was used to eliminate Ricci)

To separate evolution of k from the system, introduce new variables

$$(2) \quad k = k_i^i, \quad A_{ij} = k_{ij} - \frac{1}{3} h_{ij} k$$

Decompose the system accordingly...

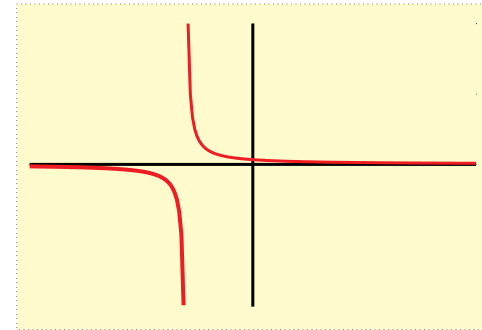
One more thing on the Trace of k_{ij}

As it follows from the ADM system,

$$(3) \quad (\partial_t - b^l D_l)k = a k^{pq} k_{pq} - D^l D_l a.$$

This can be compared to

$$(4) \quad \partial_t \psi = \psi^2$$



Question to ask: **should the lapse be chosen wisely?**

$$(5) \quad D^l D_l a = a k^{pq} k_{pq}, \quad (\text{maximal slicing}) \quad (\partial_t \psi = 0)$$

or

$$(6) \quad (\partial_t - b^l D_l)a = -a^2 k \quad (\text{harmonic slicing}) \quad (\partial_t^2 \psi = a^2 \partial_x^2 \psi + \partial_t(a\psi^2))$$

BSSN variables

Motivation: we need a decomposition for h compatible with

$$k = k_i^i, \quad A_{ij} = k_{ij} - \frac{1}{3}h_{ij}k.$$

The new variables

$\varphi = (1/12) \ln(\det h)$ is the conformal factor;

$\tilde{h}_{ij} = e^{-4\varphi}h_{ij}$, $\tilde{h}^{ij} = e^{4\varphi}h^{ij}$ are the conformal metric and its inverse;

$k = k_{pq}h^{pq}$, again, the trace of the extrinsic curvature;

$\tilde{A}_{ij} = e^{-4\varphi}A_{ij}$ the conformal analog of A_{ij} , $\tilde{A}^{ij} = e^{-4\varphi}A^{ij}$;

$\tilde{\Gamma}_j = \tilde{h}^{pq}\partial_p\tilde{h}_{qj}$, the contracted Christoffel symbol, follows from Ricci tensor decomposition;

Notice:

$$\partial\tilde{h}_{ij} = e^{-4\varphi}\left[\partial h_{ij} - \frac{1}{3}h_{ij}h^{pq}\partial h_{pq}\right].$$

BSSN formulation

Solving AMD for φ , k , \tilde{h} , \tilde{A} , $\tilde{\gamma}$ (both Hamiltonian and the momentum constraints are used)

$$(7) \quad (\partial_t - \tilde{b}^l \partial_l) \varphi = -\frac{1}{6}k + \frac{1}{6} \partial_s \tilde{b}^s$$

$$(8) \quad (\partial_t - \tilde{b}^l \partial_l) k = -D^l D_l a + \dots$$

$$(9) \quad (\partial_t - \tilde{b}^l \partial_l) \tilde{h}_{ij} = -2a \tilde{A}_{ij} + 2\tilde{h}_{s(i} \partial_{j)} \tilde{b}^s - \frac{2}{3} \tilde{h}_{ij} \partial_s \tilde{b}^s$$

$$(10) \quad (\partial_t - \tilde{b}^l \partial_l) \tilde{A}_{ij} = e^{-4\varphi} \left[\frac{1}{2} a \partial^l \partial_l \tilde{h}_{ij} + a \partial_{(i} \tilde{\Gamma}_{j)} - 2a \partial_i \partial_j \varphi - \partial_i \partial_j a \right. \\ \left. - 2a \tilde{h}_{ij} \partial^l \partial_l \varphi + \frac{1}{3} \tilde{h}_{ij} \partial^l \partial_l a \right] + \dots$$

$$(11) \quad (\partial_t - \tilde{b}^l \partial_l) \tilde{\Gamma}_i = -\frac{4}{3} a \partial_i k + \frac{1}{3} \partial_i \partial_s \tilde{b}^s + h_{si} \partial^l \partial_l \tilde{b}^s + \dots$$

Plus

$$(12) \quad (\partial_t - \tilde{b}^l \partial_l) a = -a^2 k$$

(here $\tilde{b}_j = e^{-4\varphi} b_j$; dots stand for terms vanishing in near-flat linearization)

Constraints

The analog of momentum constraint

$$(13) \quad \tilde{D}^p A_{iq} - \frac{2}{3} \partial_i k = 0;$$

The analog of Hamiltonian constraint

$$(14) \quad \tilde{R}_i^i - 8\tilde{D}^i \tilde{D}_i \varphi - 8(\partial^i \varphi)(\partial_i \varphi) + \frac{2}{3} k^2 - A^{ij} A_{ij} = 0;$$

An artificial constraint on $\tilde{\Gamma}_j$

$$(15) \quad \tilde{\Gamma}_j = \tilde{h}^{pq} \partial_p \tilde{h}_{qj}.$$

Linearization around flat space

Space-time metric is a perturbation of the Minkowski metric:

$$a = 1 + \alpha, \quad b_i = \beta_i, \quad h_{ij} = \delta_{ij} + \gamma_{ij}, \quad k_{ij} = \kappa_{ij}, \quad \alpha, \beta_i, \gamma_{ij}, \kappa_{ij} \approx O(\epsilon)$$

Then (up to higher order in perturbations)

$$\varphi = \frac{1}{12}\gamma_l^l;$$

$$k = \kappa := \kappa_l^l;$$

$$e^{-4\varphi} = 1 - \frac{1}{3}\gamma_l^l \quad \text{and} \quad e^{4\varphi} = 1 + \frac{1}{3}\gamma_l^l;$$

$$\tilde{h}_{ij} = \delta_{ij} + \gamma_{ij} - \frac{1}{3}\delta_{ij}\gamma_l^l := \delta_{ij} + \tilde{\gamma}_{ij}$$

$$\tilde{A}_{ij} = A_{ij} := \kappa_{ij} - \frac{1}{3}\delta_{ij}\kappa;$$

$$\tilde{\Gamma}_j = \Gamma_j = \partial^l \tilde{\gamma}_{lj};$$

Linearized BSSN

Evolution equations

$$(E1) \quad \partial_t \varphi = -\frac{1}{6}\kappa + \frac{1}{6}\partial^s \beta_s;$$

$$(E2) \quad \partial_t \kappa = -\partial^l \partial_l \alpha;$$

$$(E3) \quad \partial_t \alpha = -\kappa;$$

$$(E4) \quad \partial_t \tilde{\gamma}_{ij} = -2\tilde{A}_{ij} + 2\partial_{(i}\beta_{j)} - \frac{2}{3}\delta_{ij}\partial^s \beta_s;$$

$$(E5) \quad \partial_t A_{ij} = -\frac{1}{2}\partial^l \partial_l \tilde{\gamma}_{ij} + \partial_{(i}\Gamma_{j)} - 2\partial_i \partial_j \varphi - 2\delta_{ij}\partial^l \partial_l \varphi - \partial_i \partial_j \alpha + \frac{1}{3}\delta_{ij}\partial^l \partial_l \alpha;$$

$$(E6) \quad \partial_t \Gamma_i = -\frac{4}{3}\partial_i \kappa + \frac{1}{3}\partial_i \partial^s \beta_s + \partial^l \partial_l \beta_i;$$

Constraint equations:

$$(H) \quad \partial^p \partial^q \tilde{\gamma}_{pq} - 8\partial^l \partial_l \varphi = 0, \quad \text{and/or} \quad \partial^l \Gamma_l - 8\partial^l \partial_l \varphi = 0;$$

$$(M) \quad \partial^j A_{ij} - \frac{2}{3}\partial_i k = 0;$$

$$(\text{art}) \quad \Gamma_j = \partial^l \tilde{\gamma}_{lj}.$$

Boundary conditions

Unconstrained evolution problem: Given β , and $\alpha(0)$, $\varphi(0)$, $\kappa(0)$, $\tilde{\gamma}_{ij}(0)$, $A_{ij}(0)$, $\Gamma_i(0)$ plus BC solve (E1)–(E6) for α , φ , κ , $\tilde{\gamma}_{ij}$, A_{ij} , Γ_i for all times.

Boundary Conditions: (flat boundary) anything implying

$$\frac{\partial}{\partial n} A_{ij} \partial_t A^{ij} = 0, \quad \frac{\partial}{\partial n} k \partial_t k = 0.$$

Example: $A_{ij} = 0$ and $\kappa = 0$ or $\frac{\partial}{\partial n} A_{ij} = 0$ and $\frac{\partial}{\partial n} \kappa = 0$ or in combination.

Indeed, differentiating (E5) in time and using the (E1)–(E6) we get $\partial_t^2 A_{ij} = \partial^l \partial_l A_{ij}$. Use (E4) to get $\partial_t A_{ij}(0)$; retrieve $\tilde{\gamma}$ by integration. Similarly, differentiating (E2) in time and using (E3) we get $\partial_t^2 k = \partial^l \partial_l k$; $\partial_t k(0)$ is calculated from (E2), φ , α obtained by integration.

Alternatively, introduce compatible BC for $\tilde{\gamma}_{ij}$, φ , α by integration.

Evolution of the constraints

Introduce

$$\begin{aligned}M_i &= \partial^j A_{ij} - \frac{2}{3} \partial_i k; \\H1 &= \partial^p \partial^q \tilde{\gamma}_{pq} - 8 \partial^l \partial_l \varphi; \\H2 &= \partial^l \Gamma_l - 8 \partial^l \partial_l \varphi;\end{aligned}$$

Notice, that $\partial_t H1 = \partial^j M_j$ so if the momentum constraint is satisfied ($M_j = 0$) then $H1 \equiv 0$ provided it is zero initially;

$\partial_t H2 = 0$ by (E6), (E1).

So, we need to check $M_j \equiv 0$.

Evolution of M_i

The propagation of the momentum constraint is given by

$$\partial_t^2 M_j = \partial^l \partial_l M_j$$

Notice that $M_j(0) = 0$ for physical data. The value of $\partial_t M_j(0)$ can be calculated from (E5) and (E2) as (is zero for physical data)

$$\partial_t M_i(0) = \frac{1}{2} \partial^l \partial_l (\Gamma_i(0) - \partial^m \tilde{\gamma}_{im}(0)) + \frac{1}{2} \partial_i (\partial^m \Gamma_m(0) - 8 \partial^l \partial_l \varphi(0)) = 0.$$

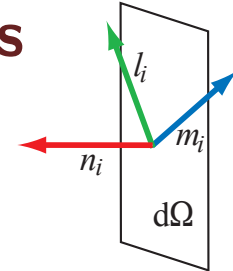
If we can find a set of well-posed boundary conditions for the linearized BSSN system that imply

$$\frac{\partial}{\partial n} M_j \partial_t M^j = 0$$

then we are done!

Constraint preserving boundary conditions

Two sets of conditions:



$$A_{ij}n^i m^j = 0, \quad A_{ij}n^i l^j = 0, \quad A_{ij}m^i l^j = 0,$$

$$A_{ij}(l^i l^j - m^i m^j) = 0, \quad A_{ij}(2n^i n^j - l^i l^j - m^i m^j) = 0, \quad \kappa = 0.$$

and

$$\frac{\partial}{\partial n} A_{ij}n^i m^j = 0, \quad \frac{\partial}{\partial n} A_{ij}n^i l^j = 0, \quad \frac{\partial}{\partial n} A_{ij}m^i l^j = 0,$$

$$\frac{\partial}{\partial n} A_{ij}(l^i l^j - m^i m^j) = 0, \quad \frac{\partial}{\partial n} A_{ij}(2n^i n^j - l^i l^j - m^i m^j) = 0, \quad \frac{\partial}{\partial n} \kappa = 0.$$

or a combination of two will work!

How to see it?

This technology uses directional derivatives, plane waves, equations

$$\partial_i = \frac{\partial}{\partial n} n_i + \frac{\partial}{\partial l} l_i + \frac{\partial}{\partial m} m_i; \quad \partial^i \partial_i = \frac{\partial}{\partial n} \frac{\partial}{\partial n} + \frac{\partial}{\partial l} \frac{\partial}{\partial l} + \frac{\partial}{\partial m} \frac{\partial}{\partial m};$$

How to see it?

At each point of the boundary A can be spanned by an orthogonal basis

$$\begin{aligned} A1 &= A_{ij}n^i m^j, & A2 &= A_{ij}n^i l^j, & A3 &= A_{ij}m^i l^j, \\ A4 &= A_{ij}(l^i l^j - m^i m^j), & A5 &= A_{ij}(2n^i n^j - l^i l^j - m^i m^j) \end{aligned}$$

In terms of directional derivatives

$$\begin{aligned} M^j \frac{\partial}{\partial n} M_j &= \left[\frac{1}{2} \frac{\partial}{\partial m} A1 + \frac{1}{2} \frac{\partial}{\partial l} A2 + 2 \frac{\partial}{\partial n} A5 - \frac{3}{2} \frac{\partial}{\partial n} \kappa \right] \\ &\quad \times \left[\frac{1}{2} \frac{\partial}{\partial m} \frac{\partial}{\partial n} A1 + \frac{1}{2} \frac{\partial}{\partial l} \frac{\partial}{\partial n} A2 + 2 \frac{\partial}{\partial n} \frac{\partial}{\partial n} A5 - \frac{3}{2} \frac{\partial}{\partial n} \frac{\partial}{\partial n} \kappa \right] \\ &+ \left[\frac{1}{2} \frac{\partial}{\partial n} A1 + \frac{1}{2} \frac{\partial}{\partial l} A3 - \frac{\partial}{\partial m} A4 - \frac{\partial}{\partial m} A5 - \frac{3}{2} \frac{\partial}{\partial m} \kappa \right] \\ &\quad \times \left[\frac{1}{2} \frac{\partial}{\partial n} \frac{\partial}{\partial n} A1 + \frac{1}{2} \frac{\partial}{\partial l} \frac{\partial}{\partial n} A3 - \frac{\partial}{\partial m} \frac{\partial}{\partial n} A4 - \frac{\partial}{\partial m} \frac{\partial}{\partial n} A5 - \frac{3}{2} \frac{\partial}{\partial m} \frac{\partial}{\partial n} \kappa \right] \\ &+ \left[\frac{1}{2} \frac{\partial}{\partial n} A2 + \frac{1}{2} \frac{\partial}{\partial m} A3 + \frac{\partial}{\partial l} A4 - \frac{\partial}{\partial l} A5 - \frac{3}{2} \frac{\partial}{\partial l} \kappa \right] \\ &\quad \times \left[\frac{1}{2} \frac{\partial}{\partial n} \frac{\partial}{\partial n} A2 + \frac{1}{2} \frac{\partial}{\partial m} \frac{\partial}{\partial n} A3 + \frac{\partial}{\partial l} \frac{\partial}{\partial n} A4 - \frac{\partial}{\partial l} \frac{\partial}{\partial n} A5 - \frac{3}{2} \frac{\partial}{\partial l} \frac{\partial}{\partial n} \kappa \right] = 0? \end{aligned}$$

Conclusions

- Nonlinear analog;
- Non-homogeneous BC;
- Curved boundary;
- Sommerfeld type BC;
- Some calculations;